THERMAL BENDING OF SANDWICH PANELS UNDER UNIAXIAL LOADING

By

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LIST OF SYMBOLS

a - Length of panel in x-direction

a_{ij} - Elements of the determinant Δ_j

b - Length of panel in y-direction

b_{ij} - Quantity defined in equations (3.34)

B_{oj}, B_{cj}, B_{ej}, B_{gj} - Fourier coefficients defined in equations (3.27)

c_{ji} - Quantity defined in equations (3.34)

C_i - Function defined in equations (3.22)

C_{oj}, C_{gj} - Fourier coefficients defined in equations (3.27)

D - Differential operator $d/d\xi$

D - Plate modulus

D_{ji} - Determinant formed from Δ_j (see p. 63)

e_x, e_{xy}, e_y - Linear components of strain

E - Young's modulus

f_o, f_1 - Auxiliary functions

F_i - Function defined in equations (3.23)

F_{ej}, F_{gj} - Fourier coefficients defined in equations (3.27)

$\bar{g}$ - $\bar{G}_x/\bar{G}_y$

g_o, g_1 - Auxiliary functions

$\bar{G}_x$, $\bar{G}_y$ - Orthotropic shear moduli of core

h', h'' - Distances to "neutral" surface

h_o, h_j - Auxiliary functions

H_{ij} - Fourier coefficient

H_x, H_{xy}, H_y - Couples formed from resolution of membrane forces

H_T - Thermal couple formed from resolution of thermal membrane forces

J_i - Function defined in equations (3.20)

k_{le} - Dimensionless edge loading parameter $-{}_oN_y^*/P_e$

K_{ij} - Fourier coefficient

m - $E't'/E''t''$

M_x, M_{xy}, M_y - Bending moments in faces

M_T - Thermal bending moment in faces

M_x^*, M_{xy}^*, M_y^* - Bending moments applied on edges of faces

N_x, N_{xy}, N_y - Additional membrane forces relative to state of initial stress

N_T - Thermal membrane force

$\tilde{N}_x$, $\tilde{N}_{xy}$, $\tilde{N}_y$ - Resultants of additional membrane forces

$\tilde{N}_T$ - Resultant of thermal membrane forces

${}_oN_x$, ${}_oN_{xy}$, ${}_oN_y$ - Initial membrane forces

${}_oN_x^*$, ${}_oN_{xy}^*$, ${}_oN_y^*$ - Applied initial edge loads

${}_o\tilde{N}_x$, ${}_o\tilde{N}_{xy}$, ${}_o\tilde{N}_y$ - Resultants of initial membrane forces

$_{T}N_{x}$, $_{T}N_{xy}$, $_{T}N_{y}$ - Total membrane forces, defined in equations (2.18)

P_e - Buckling load of a simply supported, infinitely long sandwich strip with a rigid core

P_f - Sum of buckling loads of each face of a simply supported, infinitely long sandwich strip with a rigid core

q - Distributed transverse load acting on faces

$\bar{Q}_x$, $\bar{Q}_y$ - Shear force resultants in core

Q_x^*, Q_y^* - Vertical shear forces applied on edges of faces

r_e - Dimensionless parameter

R - Dimensionless temperature gradient parameter defined in equation (3.13)

s' - $\frac{1}{2}(\bar{t} + t')$

s'' - $-\frac{1}{2}(\bar{t} + t'')$

S_i - Function defined in equations (3.21)

S_x, S_{xy}, S_y - Components of initial stress

t', t'' - Face thicknesses

$\bar{t}$ - Core thickness

$\hat{t}$ - $\frac{1}{t}(\bar{t} + \frac{t'}{2} + \frac{t''}{2})$

t_x, t_{xy}, t_y - Components of stress relative to initial state of stress

T - Change in temperature from a constant initial temperature

T_1, T_2 - Coefficients of the temperature distribution function

u, v - Displacements of middle surface of upper face relative to "neutral" surface, defined in equations (3.1)

$\tilde{u}$, $\tilde{v}$ - Displacements of "neutral" surface, defined in equations (3.1)

$\bar{u}$, $\bar{v}$ - x- and y-components of displacement in core

$\hat{u}'$, $\hat{u}''$, $\hat{v}'$, $\hat{v}''$ - x- and y- components of displacement in the faces

u_o, v_o - Functions defined in equations (3.19)

U_1, U_2, V_1, V_2 - Coefficients of displacement distribution functions in the core

V_M - Virtual work of edge moments M_x^*, M_{xy}^*, M_y^*

V_N - Virtual work of edge forces ${}_oN_x^*$, ${}_oN_{xy}^*$, ${}_oN_y^*$

V_q - Virtual work of transverse loading q

V_Q - Virtual work of edge shear forces Q_x^*, Q_y^*

w - z-component of displacement

$\hat{w}$ - Dimensionless deflection parameter

w_o - Function defined in equations (3.19)

W_c - Strain energy of core

W_f - Strain energy of faces

X, Y, Z - Solutions to ordinary differential equations; see equations (3.25)

X_c, Y_c, Z_c - Complementary solutions

X_p, Y_p, Z_p - Particular solutions

α - Coefficient of linear thermal expansion

α - Dimensionless parameter $2\bar{t}/b$

β - Aspect ratio of panel b/a

Δ_j - Determinant

ε_x, ε_{xy}, ε_y - Complete components of strain

ζ - z - s

η - y/b

θ_j - Quantity in cubic equation

λ_j - Coefficient of exponent in complementary solution; see equations (3.33)

Λ_j - λ_j^2

μ - Poisson's ratio

ξ - x/a

ρ - Dimensionless parameter $\frac{1 + m}{2} \hat{t}$

ω_x, ω_y, ω_z - Components of rotation

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THERMAL BENDING OF SANDWICH PANELS
UNDER UNIAXIAL LOADING

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The deflections of a rectangular sandwich panel subject to a thermal gradient, transverse load, and combined edge loading were calculated. Two opposite edges of the panel contained rigid inserts and were simply supported. The other two edges were simply supported, clamped, or free. Small deflection theory was assumed, as well as an orthotropic weak core. The effects of bending in the faces were also included. The five coupled, linear partial differential equations governing the deflections were transformed into two sets of equations, one of which is the well-known equation of plane stress. The remaining set was simplified to neglect all edge loading except a uniaxial force and was

solved by the use of Levy-type infinite series, which reduce the equations to ordinary differential equations with constant coefficients. The convergence of these series was investigated. A standard solution of these equations was obtained, and the resulting expressions for the displacements were written in closed form.

Values obtained from the solutions were compared with experimental values reported in the literature, and agreement was found to be good, although shortcomings of the experimental values are mentioned.

The variation of the deflections and the buckling loads for panels with varying elastic properties, dimensions, and edge loading were calculated on the University of Florida IBM 709 computer, and these data are presented in graphs. The values of the buckling loads were found to be somewhat less than those reported in a less exact analysis. It was noted that, in general, a reversal of deflections took place and that buckling modes were characterized by a large number of half-waves.

CHAPTER I

INTRODUCTION

Historical Background

Legend has it that the term "sandwich" arose in 1792 when the Fourth Earl of Sandwich devised a convenient way to have his lunch without interfering with his daily card game. Since that time the term sandwich has come to mean more than merely a way to eat. The term applies to any object constructed like a sandwich, that is, composed of three layers, the middle one being of a different material than the outer layers. Sandwich construction is a type of composition that has recently begun to achieve wide application. The construction is characterized by two thin outer layers known as the faces and a thicker inner layer known as the core. The faces are made of a stiff dense material and resist bending and membrane stresses, while the core is made of a light material offering resistance mainly to transverse shear and compression.

The original, and still most widespread, application of sandwich construction is in air and space vehicles; its

high strength-to-weight ratio is well suited for this purpose. One of the earliest applications was for the skin of the World War II DeHaviland Mosquito bomber, in which birchwood faces and a solid balsa core were used.[1] At the present time the faces are frequently made of steel or aluminum with a core made of either an expanded cellular material or a corrugated metallic sheet.

Sandwich construction is also found in curtain wall panels for office buildings;[2] with a foam plastic core it is used in refrigerator construction; it is also suitable as a hull material for boats when a balsa core is used.

Because of this wide use of sandwich construction, especially in panel form, the elastic analysis of a sandwich plate is a timely and important problem. The nature of the core necessitates the consideration of several effects not present in the usual analyses of thin plates. The strength of corrugated cores will obviously depend on the direction of the corrugations. Cores made of expanded materials are constructed of thin strips of material which run in one direction. The properties of either type of core are thus orthotropic. Because of the relative thickness of the core, the effect of shearing deflection is important. Since cores contain air spaces, and thus act as thermal insulators,

sandwich construction used in an air vehicle will be subject to a temperature difference between the two faces. Thermal stresses arise from this temperature gradient.

The earliest study of a sandwich plate was performed by Gough, Elam, and DeBruyne;[3] they considered the problem of a thin sheet supported by a continuous elastic foundation in order to study "wrinkling," or instability of the faces characterized by a large number of buckling waves. Later, Williams, Leggett, and Hopkins[4] and Leggett and Hopkins[5] examined the Euler type of buckling of sandwich plates as a whole; they accounted for shearing deformations in the core by using the "tilting method," as termed by March,[6] in which a line originally normal to the middle surface tilts during deformation through an arbitrary angle. Use of an approximate strain energy method determined the buckling load. March and others extended this method to include the effects of orthotropic cores and orthotropic faces of differing thicknesses and elastic properties.[7,8,9] Libove and Batdorf developed a method of analysis in which the sandwich plate is treated as an orthotropic plate with the effect of shearing deformation included.[10] Reissner developed differential equations for the finite deflection of sandwich plates.[11]

Hoff used the assumption of a weak core which resists only vertical shear and the principle of virtual displacements to derive exact equations for the bending of a sandwich plate.[12] He was able to solve the problem for a simply supported plate by assuming that rigid inserts were set into the core edges. In this way the boundary conditions were altered to permit a solution.

In 1951 Eringen used the principle of virtual work to derive extremely general equations which account for the effects of compression and flexure of the core.[13] In 1952 Gerard, by an order of magnitude analysis, examined the limits of validity of the simplifications commonly assumed in sandwich plate analysis.[14] Goodier and Hsu considered nonsinusoidal buckling modes.[15] In 1959 Chang and Ebcioglu extended Hoff's theory to include an orthotropic core and dissimilar faces and found the buckling load for a simply supported plate.[16] Recently, Yu, in a series of papers, has included the effects of rotatory inertia and shear in the faces.[17,18,19,20]

Thermal stresses in sandwich plates were treated by Bijlaard, who used a homogeneous plate approximation.[21] Chang and Ebcioglu extended their approach to include the effect of a constant temperature gradient,[22] and Ebcioglu

later solved the problem for a temperature gradient which varies arbitrarily across the area of the plate.[23] Utilizing Hoff's approximation for the boundary conditions, he presented a solution for the bending of a plate simply supported on all edges. Ebcioglu et al. also give experimental results for the deflection due to a thermal gradient.[24]

Definition of the Problem

It is the aim of the present analysis (1) to extend Ebcioglu's work in order to develop equations which include the effects of bending of the faces, a linear temperature distribution through the thickness, and various boundary conditions, (2) to solve the equations for the exact boundary conditions rather than making Hoff's approximations, and (3) to present graphs illustrating the effects of various parameters of the plate on the deflection and buckling load.

A Cartesian coordinate system is defined for a rectangular sandwich plate as shown in Figure 1. The middle plane of the core lies in the xy-plane, with two adjacent edges lying along the x- and y-axes; the z-axis points downward. In the notational system employed here a single prime denotes quantities related to the lower face, a double prime those related to the upper face, and a superposed bar those related to the core.

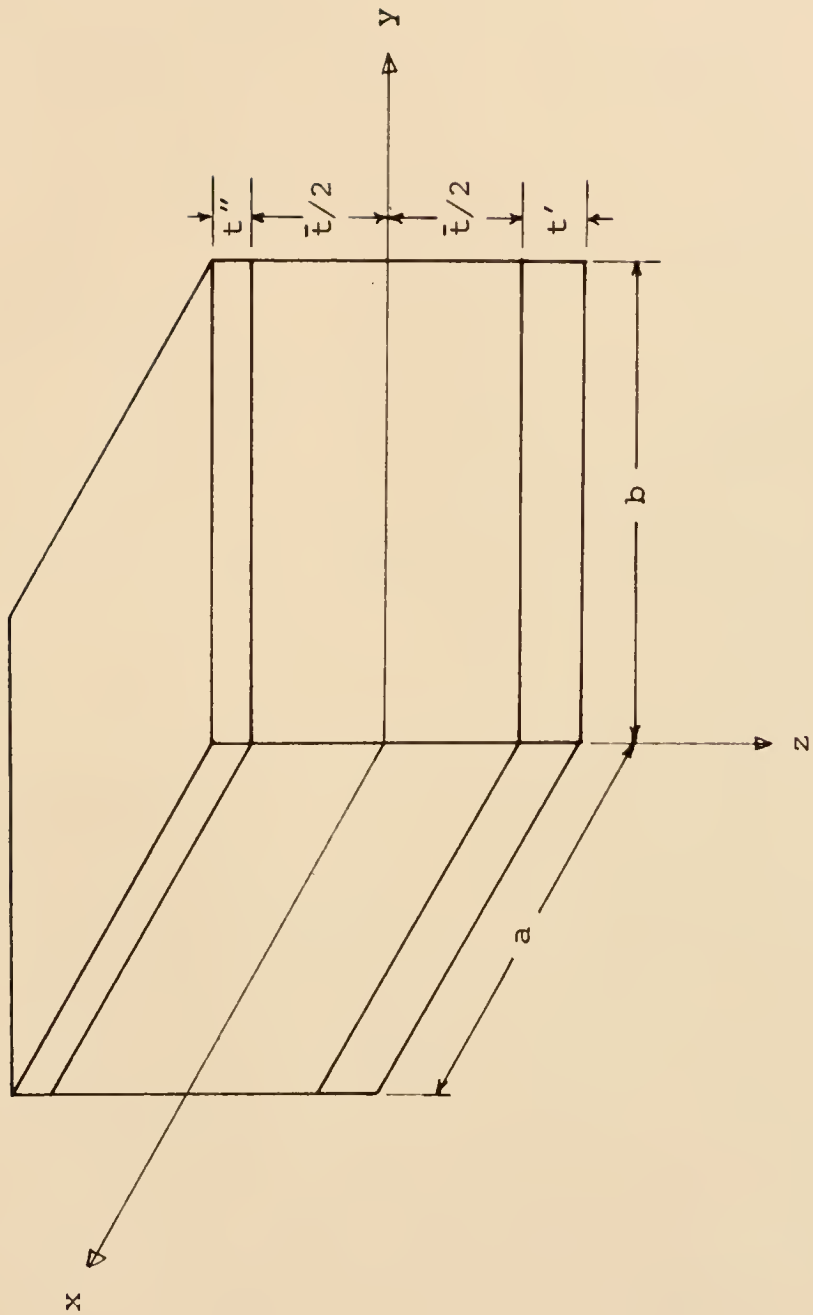

Figure 1. Dimensions of the rectangular sandwich panel

The plate has lateral dimensions a and b along the x- and y-directions respectively, and the thicknesses of the upper face, core, and lower face are t'', $\bar{t}$, and t' respectively. The faces may differ in Young's modulus E, but have the same Poisson ratio μ. The core is assumed to be homogeneous and orthotropic.

The temperature change T of each face is defined as the change from some constant initial reference temperature and is arbitrarily distributed in the x- and y-directions; the variation in the z-direction through the thickness of each face is given by the following equations:

$$\begin{aligned} T' &= T_1'\zeta + T_2' \\ T'' &= T_1''\zeta + T_2'' \end{aligned} \tag{1.1}$$

where T_1', T_1'', T_2', and T_2'' are functions of x and y only;

$$\zeta = z - s \tag{1.2}$$

where

$$s = s' = \frac{1}{2}(\bar{t} + t') \text{ for the lower face}$$

$$s = s'' = -\frac{1}{2}(\bar{t} + t'') \text{ for the upper face.}$$

Arbitrary transverse pressures q' and q'' act normal to the outer surface of each face, while on the edges of

each face there act applied axial forces ${}_{o}N_{x}^{*}$, ${}_{o}N_{xy}^{*}$, and ${}_{o}N_{y}^{*}$, applied moments M_{x}^{*}, M_{xy}^{*}, and M_{y}^{*}, and applied vertical shear forces Q_{x}^{*} and Q_{y}^{*}, where each of these quantities is defined per unit length of the boundary. The subscripts follow the usual convention for stress notation. These forces are uniformly distributed in the z-direction on each face and do not act on the core, but on any edge where external loading is applied a rigid insert is set into the core for its protection. A more complete description of the insert is given in the section on boundary conditions.

The Kirchoff hypothesis is assumed to hold for the deflections of the faces. The temperature is considered to be in steady-state equilibrium and the effect of deformation on temperature is neglected. Account is taken of the fact that elastic properties vary with temperature by permitting each face to have a different value of E. The "weak core" assumption is made, that is, the core is considered to be so weak in its resistance to bending and axial stresses that only the antiplane stresses $\bar{\tau}_{zx}$, $\bar{\tau}_{zy}$, and $\bar{\tau}_{zz}$ are present. The core is considered incompressible in the z-direction. It is assumed that neither local buckling of the faces nor bond failure between the faces and the core occurs.

CHAPTER II

DERIVATION OF THE EQUATIONS

The Strain Energy Expression

A macroscopic approach has been chosen for the derivation of the differential equations governing the five independent components of displacement. It is felt that this approach, based on the theorem of virtual work, more clearly shows the assumptions involved in formulating a mathematical model which adheres to physical reality yet allows a solution; this approach also permits a more straightforward derivation of the boundary conditions than would the alternate approach based on the microscopic equilibrium of an element of the plate.

The theorem of virtual work states that if an elastic body is in equilibrium, the variation of the strain energy equals the virtual work of external forces acting on the body. The expression for strain energy for each layer of the sandwich plate is written and the variation taken by applying the variational calculus.

The derivation of the strain energy expressions for the faces proceeds from the consideration that they are initially in a state of equilibrium under the action of large membrane stresses. From this initial state the faces undergo additional deformations. The approach used will be based on that of Timoshenko[25] and Herrmann.[26]

The initial components of stress are denoted by S_x, S_y, and S_{xy}. They are assumed not to vary through the thickness of each face, and to be of an order of magnitude higher than the stresses produced during the additional deformation. The following initial stress resultants are defined:

$$
\begin{aligned}
{}_oN_x &= tS_x \\
{}_oN_y &= tS_y \\
{}_oN_{xy} &= tS_{xy}
\end{aligned}
\tag{2.1}
$$

Since the plate is initially in equilibrium, the following equilibrium equations may be written for each face:

$$
\begin{aligned}
\frac{\partial\, {}_oN_x}{\partial x} + \frac{\partial\, {}_oN_{xy}}{\partial y} &= 0 \\
\frac{\partial\, {}_oN_{xy}}{\partial x} + \frac{\partial\, {}_oN_y}{\partial y} &= 0
\end{aligned}
\tag{2.2}
$$

The boundary conditions for the initial state are

$$\begin{aligned} {}_{o}N_{y}^{*} &= {}_{o}N_{y} \\ {}_{o}N_{xy}^{*} &= {}_{o}N_{xy} \end{aligned} \tag{2.3a}$$

on $y = 0$, $y = b$, and

$$\begin{aligned} {}_{o}N_{x}^{*} &= {}_{o}N_{x} \\ {}_{o}N_{xy}^{*} &= {}_{o}N_{xy} \end{aligned} \tag{2.3b}$$

on $x = 0$, $x = a$, where ${}_{o}N_{x}^{*}$, ${}_{o}N_{y}^{*}$, and ${}_{o}N_{xy}^{*}$ are the initially applied loads on the edges of each face.

The additional components of strain are assumed small compared to unity; it is further assumed that the elongation and shear terms are of the same order and the rotation terms are of a smaller order. However, since products of initial forces and rotation terms will occur, these latter terms will be retained. Then the components of strain relative to the initial state are

$$\varepsilon_{x} \approx e_{x} + \frac{1}{2}(\omega_{z}^{2} + \omega_{y}^{2}) \tag{2.4a}$$

$$\varepsilon_{y} \approx e_{y} + \frac{1}{2}(\omega_{x}^{2} + \omega_{z}^{2}) \tag{2.4b}$$

$$\varepsilon_{z} \approx e_{z} + \frac{1}{2}(\omega_{y}^{2} + \omega_{x}^{2}) \tag{2.4c}$$

$$\varepsilon_{xy} \approx e_{xy} - \frac{1}{2}\omega_x \omega_y \tag{2.4d}$$

$$\varepsilon_{yz} \approx e_{yz} - \frac{1}{2}\omega_y \omega_z \tag{2.4e}$$

$$\varepsilon_{zx} \approx e_{zx} - \frac{1}{2}\omega_z \omega_x \tag{2.4f}$$

where e_x, e_y, e_z, e_{xy}, e_{yz}, and e_{zx} are the linear strain components and ω_x, ω_y, and ω_z are the rotation components.

Based on the Kirchoff hypothesis, the deflections in the faces are given by

$$\hat{u} = u - \zeta \frac{\partial w}{\partial x}$$
$$\hat{v} = v - \zeta \frac{\partial w}{\partial y} \tag{2.5}$$

where u and v represent the displacement components due to the stretching of the middle surface in each face from the initial equilibrium state, and ζ is given by equation (1.2).

The rotations ω_z are considered to be negligible for plate deformations. Then the use of equations (2.5) in equations (2.4) produces the strain-displacement relations:

$$\varepsilon_x \approx \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \zeta \frac{\partial^2 w}{\partial x^2} \tag{2.6a}$$

$$\varepsilon_y \approx \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - \zeta \frac{\partial^2 w}{\partial y^2} \tag{2.6b}$$

$$\varepsilon_{xy} \approx \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - 2\zeta\frac{\partial^2 w}{\partial x \partial y}\right) \tag{2.6c}$$

$$\varepsilon_z \approx \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \tag{2.6d}$$

$$\varepsilon_{xz} \approx 0 \tag{2.6e}$$

$$\varepsilon_{yz} \approx 0 \tag{2.6f}$$

The additional components of stress t_x, t_y, and t_{xy} are assumed to be related to the additional strains by the Duhamel-Neuman equations. The error introduced by this assumption is of the order of the ratio of the initial stress to the elastic modulus of the face.[26] Furthermore, the transverse normal stress σ_{zz} will be neglected in accordance with classical plate theory. Therefore, the following equations for plane stress[27] will apply:

$$t_x = \frac{E}{1-\mu^2}(\varepsilon_x + \mu\varepsilon_y) - \frac{E\alpha T}{1-\mu}$$

$$t_y = \frac{E}{1-\mu^2}(\varepsilon_y + \mu\varepsilon_x) - \frac{E\alpha T}{1-\mu}$$

$$t_{xy} = \frac{E}{1+\mu}\varepsilon_{xy}$$

where α is the coefficient of linear thermal expansion.

The following stress resultants are defined in the faces. The membrane forces per unit length are

$$N_x = \int_{-t/2}^{t/2} t_x d\zeta = \frac{Et}{1-\mu^2}\left[\frac{\partial u}{\partial x} + \mu\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\mu\left(\frac{\partial w}{\partial y}\right)^2\right] - N_T$$

$$N_y = \int_{-t/2}^{t/2} t_y d\zeta = \frac{Et}{1-\mu^2}\left[\frac{\partial v}{\partial y} + \mu\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{1}{2}\mu\left(\frac{\partial w}{\partial x}\right)^2\right] - N_T \tag{2.7}$$

$$N_{xy} = \int_{-t/2}^{t/2} t_{xy}\, d\zeta = \frac{Et}{2(1+\mu)}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]$$

where the thermal force per unit length is

$$N_T = \frac{E\alpha}{1-\mu}\int_{-t/2}^{t/2} T d\zeta = \frac{E\alpha T_2 t}{1-\mu}$$

The bending moments per unit length are

$$M_x = \int_{-t/2}^{t/2} \zeta t_x d\zeta = -D\left[\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right] - M_T$$

$$M_y = \int_{-t/2}^{t/2} \zeta t_y d\zeta = -D\left[\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right] - M_T \tag{2.8}$$

$$M_{xy} = \int_{-t/2}^{t/2} \zeta t_{xy} d\zeta = -D(1-\mu)\frac{\partial^2 w}{\partial x \partial y}$$

where the thermal moment M_T is

$$M_T = \alpha D(1 + \mu)T,$$

and the plate modulus is

$$D = \frac{Et^3}{12(1 - \mu^2)}$$

The strain energy W_F in the faces is given by the expression[27]

$$W_F = \frac{1}{2}\int_V \left[t_x\varepsilon_x + t_y\varepsilon_y + 2t_{xy}\varepsilon_{xy} - 2\alpha T(t_x + t_y)\right]dV$$

$$+ \int_V (S_x\varepsilon_x + S_y\varepsilon_y + 2S_{xy}\varepsilon_{xy})dV \qquad (2.9)$$

and the integration is extended over the volume of the body in the initial state.

The stress resultants (2.1), (2.7), and (2.8), together with the strains (2.7) and the temperature distribution (1.1), are substituted into expression (2.9). After integration through the thickness, there results

$$W_F = \frac{D}{2}\int_0^b \int_0^a \left\{(\nabla^2 w)^2 - 2(1 - \mu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right.$$

$$\left. - 2(1 + \mu)\alpha T, \nabla^2 w\right\}dxdy +$$

$$+ \int_0^b \int_0^a \left\{ \left({}_oN_x + \frac{1}{2}N_x \right) \left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \right] \right.$$

$$+ \left({}_oN_y + \frac{1}{2}N_y \right) \left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \right]$$

$$+ \left({}_oN_{xy} + \frac{1}{2}N_{xy} \right) \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \right]$$

$$\left. - \alpha T_1 (N_x + N_y) \right\} dxdy \qquad (2.10)$$

The first integral in (2.10) represents the strain energy of bending, while the second represents strain energy of the stretching of the middle surface.

Since a "weak" core that is inextensible in the z-direction is assumed, only the shear stresses $\bar{\tau}_{zx}$ and $\bar{\tau}_{zy}$ exist. If the rotation terms ω_z are neglected, the orthotropic stress-strain-displacement relations are

$$\bar{\tau}_{zx} = 2\bar{G}_x \bar{\varepsilon}_{zx} = \bar{G}_x \left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial w}{\partial x} \right)$$

$$\bar{\tau}_{zy} = 2\bar{G}_y \bar{\varepsilon}_{zy} = \bar{G}_y \left(\frac{\partial \bar{v}}{\partial z} + \frac{\partial w}{\partial y} \right) \qquad (2.11)$$

where $\bar{G}_x$ and $\bar{G}_y$ are the orthotropic shear moduli.

It follows from equations (2.11) and from the assumption that $\bar{\varepsilon}_z = 0$ that the displacements in the core are given by the relations

$$\bar{u} = U_1 z + U_2$$

$$\bar{v} = V_1 z + V_2 \tag{2.12}$$

where the functions $U_1(x,y)$, $U_2(x,y)$, $V_1(x,y)$, and $V_2(x,y)$ are found from the conditions of continuity of displacements at the interfaces between the core and the faces. In this way

$$U_1 = \frac{u' - u''}{\bar{t}} + \frac{t' + t''}{2\bar{t}} \frac{\partial w}{\partial x}$$

$$U_2 = \frac{u' + u''}{2} + \frac{t' - t''}{4} \frac{\partial w}{\partial x}$$

$$V_1 = \frac{v' - v''}{\bar{t}} + \frac{t' + t''}{2\bar{t}} \frac{\partial w}{\partial y}$$

$$V_2 = \frac{v' + v''}{2} + \frac{t' - t''}{4} \frac{\partial w}{\partial y}$$

The following stress resultants are defined in the core. The shear forces per unit length are

$$\bar{Q}_x = \int_{-\bar{t}/2}^{\bar{t}/2} \bar{\tau}_{zx}\, dz = \bar{t}\bar{G}_x \left(\frac{u' - u''}{\bar{t}} + \hat{t} \frac{\partial w}{\partial x}\right)$$

$$\bar{Q}_y = \int_{-\bar{t}/2}^{\bar{t}/2} \bar{\tau}_{zy}\, dz = \bar{t}\bar{G}_y \left(\frac{v' - v''}{\bar{t}} + \hat{t} \frac{\partial w}{\partial y}\right) \tag{2.13}$$

where

$$\hat{t} = \frac{1}{\bar{t}}\left(\bar{t} + \frac{t'}{2} + \frac{t''}{2}\right) \tag{2.14}$$

The strain energy W_c of the core is of the form

$$W_c = \int_V \left(\bar{\tau}_{zx}\bar{\varepsilon}_{zx} + \bar{\tau}_{zy}\bar{\varepsilon}_{zy}\right) dV \tag{2.15}$$

The expressions (2.12) are substituted into the orthotropic stress-strain-displacement relations (2.11) and then into the strain energy term (2.15). After integration through the core thickness, there results

$$W_c = \frac{\bar{t}}{2}\int_0^b \int_0^a \left[\bar{G}_x\left(U_1^2 + \frac{\partial w}{\partial x}\right)^2 + \bar{G}_y\left(V_1^2 + \frac{\partial w}{\partial y}\right)^2\right] dxdy \tag{2.16}$$

The Equilibrium Equations

According to the principle of virtual work,

$$\delta W_F' + \delta W_F'' + \delta W_c - \delta V_q - \delta V_N' - \delta V_N'' - \delta V_M' - \delta V_M'' -$$

$$\delta V_Q' - \delta V_Q'' = 0 \tag{2.17}$$

where the virtual work due to the applied loadings is as follows: (1) from the transverse pressure

$$\delta V_q = \int_0^b \int_0^a (q'' - q')\delta w dx dy$$

(2) from the edge forces on each face

$$\delta V_N = \int_0^b ({}_oN_x^* \delta u + {}_oN_{xy}^* \delta v)dy + \int_0^a ({}_oN_y^* \delta v + {}_oN_{xy}^* \delta u)dx$$

(3) from the edge moments on each face

$$\delta V_M = -\int_0^b M_x^* \delta(\frac{\partial w}{\partial x})dy - \int_0^a M_y^* \delta(\frac{\partial w}{\partial y})dx$$

$$+ \int_0^b \frac{\partial M_{xy}^*}{\partial y} \delta w dy + \int_0^a \frac{\partial M_{xy}^*}{\partial x} \delta w dx - 2M_{xy}^* \Big|_0^a \Big|_0^b \delta w$$

(4) from the shear forces on each face

$$\delta V_Q = \int_0^b Q_x^* \delta w dy + \int_0^a Q_y^* \delta w dy$$

In forming the variations use is made of the formulas

$$\int_0^b \int_0^a \frac{\partial F}{\partial x} dx dy = \int_0^b F(x,y) \Big|_{x=0}^{x=a} dy$$

$$\int_0^b \int_0^a \frac{\partial F}{\partial y}\, dxdy = \int_0^a F(x,y)\Big|_{y=0}^{y=b} dx$$

and of integration by parts. Equation (2.17) becomes

$$\int_0^b \int_0^a \Bigg[\Bigg\{ (D' + D'')\nabla^4 w + \nabla^2(M_T' + M_T'') - ({}_TN_x' + {}_TN_x'')\frac{\partial^2 w}{\partial x^2}$$

$$- 2({}_TN_{xy}' + {}_TN_{xy}'')\frac{\partial^2 w}{\partial x \partial y} - ({}_TN_y' + {}_TN_y'')\frac{\partial^2 w}{\partial y^2} + q' - q''$$

$$- \hat{t}\left[\bar{G}_x \left(\frac{\partial u'}{\partial x} - \frac{\partial u''}{\partial x} + \bar{t}\hat{t}\,\frac{\partial^2 w}{\partial x^2} \right) + \bar{G}_y \left(\frac{\partial v'}{\partial y} - \frac{\partial v''}{\partial y} + tt\,\frac{\partial^2 w}{\partial y^2} \right) \right] \Bigg\}\, \delta w$$

$$- \left\{ \frac{\partial\, {}_TN_x'}{\partial x} + \frac{\partial\, {}_TN_{xy}'}{\partial y} - \bar{G}_x \left(\frac{u' - u''}{\bar{t}} + \hat{t}\,\frac{\partial w}{\partial x} \right) \right\} \delta u'$$

$$- \left\{ \frac{\partial\, {}_TN_x''}{\partial x} + \frac{\partial\, {}_TN_{xy}''}{\partial y} + \bar{G}_x \left(\frac{u' - u''}{\bar{t}} + \hat{t}\,\frac{\partial w}{\partial x} \right) \right\} \delta u''$$

$$- \left\{ \frac{\partial\, {}_TN_{xy}'}{\partial x} + \frac{\partial\, {}_TN_y'}{\partial y} - \bar{G}_y \left(\frac{v' - v''}{\bar{t}} + \hat{t}\,\frac{\partial w}{\partial y} \right) \right\} \delta v'$$

$$- \left\{ \frac{\partial\, {}_TN_{xy}''}{\partial x} + \frac{\partial\, {}_TN_y''}{\partial y} + \bar{G}_y \left(\frac{v' - v''}{\bar{t}} + \hat{t}\,\frac{\partial w}{\partial y} \right) \right\} \delta v'' \Bigg] dxdy$$

$$+ \int_0^b \Bigg[(-M_x + M_x^*)\,\delta\left(\frac{\partial w}{\partial x}\right) + \left(\frac{\partial M_x}{\partial x} + 2\,\frac{\partial M_{xy}}{\partial y} + {}_TN_x \frac{\partial w}{\partial x} + {}_TN_{xy}\frac{\partial w}{\partial y} \right.$$

$$\left. + \hat{t}\bar{Q}_x - \frac{\partial M_{xy}^*}{\partial y} - Q_x^* \right) \delta w + ({}_TN_x - {}_oN_x^*)\,\delta u$$

$$+ ({}_TN_{xy} - {}_oN_{xy}^*)\,\delta v \Bigg] dy +$$

$$+ \int_o^a \left[(-M_y + M_y^*)\delta\left(\frac{\partial w}{\partial y}\right) + \left(\frac{\partial M_y}{\partial y} + 2\frac{\partial M_{xy}}{\partial x} + {}_\tau N_y \frac{\partial w}{\partial y} + {}_\tau N_{xy}\frac{\partial w}{\partial x}\right.\right.$$

$$\left. + \hat{t}\bar{Q}_y - \frac{\partial M_{xy}^*}{\partial x} - Q_y^*\right)\delta w + ({}_\tau N_{xy} - {}_o N_{xy}^*)\delta u$$

$$\left. + ({}_\tau N_y - {}_o N_y^*)\delta v \right] dx + 2(M_{xy} - M_{xy}^*)\Big|_o^a \Big|_o^b \delta w = 0 \qquad (2.18)$$

In expression (2.18) the total membrane forces are defined by

$$\begin{aligned} {}_\tau N_x &= {}_o N_x + N_x \\ {}_\tau N_y &= {}_o N_y + N_y \\ {}_\tau N_{xy} &= {}_o N_{xy} + N_{xy} \end{aligned} \qquad (2.19)$$

The integrals extend over the body in the initial state, and virtual displacements are taken as relative to that state.

Since the virtual displacements $\delta u'$, $\delta u''$, $\delta v'$, $\delta v''$, and δw are arbitrary within the plate, their coefficients in equation (2.18) must vanish. In this manner the following equations of equilibrium result. In the first of these equations the terms $(N_x' + N_x'')\frac{\partial^2 w}{\partial x^2}$, $(N_y' + N_y'')\frac{\partial^2 w}{\partial y^2}$, and $(N_{xy}' + N_{xy}'')\frac{\partial^2 w}{\partial x \partial y}$ are neglected. The terms in parentheses are membrane stresses due to bending of the plate after initial stress and are negligible.

$$(D' + D'')\nabla^4 w - ({}_oN_x' + {}_oN_x'')\frac{\partial^2 w}{\partial x^2} - 2({}_oN_{xy}' + {}_oN_{xy}'')\frac{\partial^2 w}{\partial x \partial y}$$

$$- ({}_oN_y' + {}_oN_y'')\frac{\partial^2 w}{\partial y^2} + \nabla^2(M_T' + M_T'') - \hat{t}\Big[\bar{G}_x\Big(\frac{\partial u'}{\partial x} - \frac{\partial u''}{\partial x} + \bar{t}\hat{t}\,\frac{\partial^2 w}{\partial x^2}\Big)$$

$$+ \bar{G}_y\Big(\frac{\partial v'}{\partial y} - \frac{\partial v''}{\partial y} + \bar{t}\hat{t}\,\frac{\partial^2 w}{\partial y^2}\Big)\Big] + q' - q'' = 0 \qquad (2.20a)$$

$$\frac{\partial N_x'}{\partial x} + \frac{\partial N_{xy}'}{\partial y} - \bar{G}_x\Big(\frac{u' - u''}{\bar{t}} + \hat{t}\,\frac{\partial w}{\partial x}\Big) = 0 \qquad (2.20b)$$

$$\frac{\partial N_x''}{\partial x} + \frac{\partial N_{xy}''}{\partial y} + \bar{G}_x\Big(\frac{u' - u''}{\bar{t}} + \hat{t}\,\frac{\partial w}{\partial x}\Big) = 0 \qquad (2.20c)$$

$$\frac{\partial N_{xy}'}{\partial x} + \frac{\partial N_y'}{\partial y} - \bar{G}_y\Big(\frac{v' - v''}{\bar{t}} + \hat{t}\,\frac{\partial w}{\partial y}\Big) = 0 \qquad (2.20d)$$

$$\frac{\partial N_{xy}''}{\partial x} + \frac{\partial N_y''}{\partial y} + \bar{G}_y\Big(\frac{v' - v''}{\bar{t}} + \hat{t}\,\frac{\partial w}{\partial y}\Big) = 0 \qquad (2.20e)$$

These five equations in five unknowns reduce to those derived by Ebcioglu[23] for the case of constant temperature variation through the plate thickness and for small deflections.

Equations (2.20) can be seen to contain the equations of homogeneous thin plate theory. If the core is assumed to be rigid, i.e., if $\bar{G}_{xz} = \bar{G}_{yz} \to \infty$, then it follows from equations (2.11) that $\bar{\varepsilon}_{xz} = \bar{\varepsilon}_{yz} = 0$ and there is no contribution from the core to the total strain energy. The coefficients of $\bar{G}_x$ and $\bar{G}_y$ in equations (2.20) disappear and, except for the thermal terms, the resulting equations are the classical ones of Saint-Venant, describing two plates constrained to have a common vertical deflection and subjected to lateral and edge loads. These plates correspond in their dimensions and properties to the faces of the sandwich panel.

Saint-Venant's equations also result if the thickness $\bar{t}$ of the core is allowed to go to zero, since in this case the integral (2.15) then degenerates to zero. In either case the effect of the core is only to constrain the two faces to have a common vertical deflection. The core has no stress-resisting properties of its own, so the fact that it retains its thickness for the first case is of no importance.

It is noted that a formal application of these limiting processes to equations (2.20) is meaningless and produces incorrect results, since considerations preceding the derivation of this equation are involved.

The Boundary Conditions

The boundary conditions arise from consideration of the line integrals occurring in the variational equation (2.18). These line integrals may be made to vanish by setting either the variations or their coefficients equal to zero. The boundary conditions are linearized in the same manner as equation (2.20a). After use of equations (2.3), they are found to be: (1) on the boundaries y = 0 and y = b either

$$u' = 0 \tag{2.21a}$$

or

$$N_{xy}' = 0 \tag{2.21b}$$

either

$$u'' = 0 \tag{2.22a}$$

or

$$N_{xy}'' = 0 \tag{2.22b}$$

either

$$v' = 0 \tag{2.23a}$$

or

$$N_y' = 0 \tag{2.23b}$$

either

$$v'' = 0 \tag{2.24a}$$

or

$$N_y'' = 0 \tag{2.24b}$$

either

$$w = 0 \tag{2.25a}$$

or

$$\frac{\partial}{\partial y}(M_y' + M_y'') + 2\frac{\partial}{\partial x}(M_{xy}' + M_{xy}'') + ({}_oN_y' + {}_oN_y'')\frac{\partial w}{\partial y}$$

$$+ \hat{t}\bar{Q}_y = Q_y^{*\prime} + Q_y^{*\prime\prime} + \frac{\partial}{\partial x}(M_{xy}^{*\prime} + M_{xy}^{*\prime\prime}) \tag{2.25b}$$

either

$$\frac{\partial w}{\partial y} = 0 \tag{2.26a}$$

or

$$M_y' + M_y'' = M_y^{*\prime} + M_y^{*\prime\prime} \tag{2.26b}$$

(2) on the boundaries x = 0 and x = a

either

$$u' = 0 \tag{2.27a}$$

or

$$N_x' = 0 \tag{2.27b}$$

either

$$u'' = 0 \tag{2.28a}$$

or

$$N_x'' = 0 \tag{2.28b}$$

either

$$v' = 0 \tag{2.29a}$$

or

$$N_{xy}' = 0 \tag{2.29b}$$

either

$$v'' = 0 \tag{2.30a}$$

or

$$N_{xy}'' = 0 \tag{2.30b}$$

either

$$w = 0 \tag{2.31a}$$

or

$$\frac{\partial}{\partial x}(M_x' + M_x'') + 2\frac{\partial}{\partial y}(M_{xy}' + M_{xy}'') + ({}_oN_x' + {}_oN_x'')\frac{\partial w}{\partial x}$$

$$+ \hat{t}\bar{Q}_x = Q_x^{*\prime} + Q_x^{*\prime\prime} + \frac{\partial}{\partial y}(M_{xy}^{*\prime} + M_{xy}^{*\prime\prime}) \tag{2.31b}$$

either

$$\frac{\partial w}{\partial x} = 0 \tag{2.32a}$$

or

$$M_x' + M_x'' = M_x^{*\prime} + M_x^{*\prime\prime} \tag{2.32b}$$

(3) for the corners (0,0), (a,0), (0,b), and (a,b)

either

$$w = 0 \tag{2.33a}$$

or

$$M_{xy}\Big|_0^a\Big|_0^b = M_{xy}^*\Big|_0^a\Big|_0^b \tag{2.33b}$$

The boundary conditions for a rectangular sandwich panel simply supported along the edges perpendicular to the

y-axis and variously supported along the other two edges, containing rigid inserts set into the edges of the core that are perpendicular to the y-axis, may be selected from the preceding sets of boundary conditions.

A simply supported edge may be supported either on knife-edges or by pins of negligible cross-sectional dimensions, about which the panel may freely rotate. If the latter case is considered for a sandwich panel, the pins may be imagined as running through both faces, parallel and arbitrarily close to the edges. They restrain the boundary from displacement perpendicular to itself, but permit parallel motion. The restraint in the perpendicular direction will cause a normal stress at the boundary.

If, on the other hand, the knife-edge type of simple support is envisioned, the edge of each face is considered to be in contact with a wedge-shaped support along the line forming the vertex of the wedge. In this case the edge is free to translate in directions both parallel and perpendicular to itself.

Finally, the effect of a thin rigid insert is considered. The insert is envisioned as being set into the core so that it is flush with the edges of the upper and lower faces and welded to these faces at its top and bottom

edges. Its effect is to prevent the edges from having displacements in the direction parallel to themselves. If the insert is perfectly insulated, so that thermal effects cause no expansion of it, boundary conditions (2.21a) and (2.22a) or (2.27a) and (2.28a) will apply. If, however, thermal expansion of the insert is allowed, the boundary conditions must allow for some displacement of the edges parallel to themselves. In general, vertical displacements of the edges would also occur, but it is assumed that the supports prevent such displacements. The extreme case occurs when the insert offers no resistance to the free thermal expansion of the edges. Then boundary conditions (2.29a) and (2.30a) are replaced by

$$v' = \alpha' \int_{b/2}^{y} T_2' dy \tag{2.34a}$$

$$v'' = \alpha'' \int_{b/2}^{y} T_2'' dy \tag{2.34b}$$

and conditions (2.21a) and (2.22a) are replaced by

$$u' = \alpha' \int_{a/2}^{x} T_2' dx \tag{2.35a}$$

$$u'' = \alpha'' \int_{a/2}^{x} T_2'' dx \tag{2.35b}$$

In general, the displacement along the edge will be somewhere between zero and these values, depending on the coefficient of thermal expansion of the material of the insert. The vertical deflections at the center of the plate were calculated for both zero deflection and the extreme values in equations (2.35); only a slight difference resulted.

The problem to be examined here is a plate containing a rigid insert along the two edges parallel to the x-axis, which are simply supported on knife-edges. The two edges parallel to the y-axis may be simply supported by knife-edges, clamped, or free. Furthermore, the only external edge loading applied to the face edges is the initial load ${}_oN_y^*$, which is assumed to be constant along the edge. No transverse loading acts on the lower face ($q' = 0$).

The boundary conditions to be used are as follows.
(1) On the boundaries $y = 0$ and $y = b$

$$u' = \alpha' \int_{a/2}^{x} T_2' dx \tag{2.36a}$$

$$u'' = \alpha'' \int_{a/2}^{x} T_2'' dx \tag{2.36b}$$

$$N_y' = 0 \tag{2.36c}$$

$$N_y'' = 0 \tag{2.36d}$$

$$w = 0 \tag{2.36e}$$

$$M_y' + M_y'' = 0 \tag{2.36f}$$

where the right-hand sides of equations (2.35) should be set equal to zero if the inserts do not expand when heated.

(2) On the edges x = 0 and x = a, for the case of simply supported edges,

$$N_x' = 0 \tag{2.37a}$$

$$N_x'' = 0 \tag{2.37b}$$

$$N_{xy}' = 0 \tag{2.37c}$$

$$N_{xy}'' = 0 \tag{2.37d}$$

$$w = 0 \tag{2.37e}$$

$$M_x' + M_x'' = 0 \tag{2.37f}$$

For the case of clamped edges

$$u' = 0 \tag{2.38a}$$

$$u'' = 0 \tag{2.38b}$$

$$N_{xy}' = 0 \tag{2.38c}$$

$$N_{xy}'' = 0 \tag{2.38d}$$

$$w = 0 \tag{2.38e}$$

$$\frac{\partial w}{\partial x} = 0 \tag{2.38f}$$

For the case of free edges

$$N_x' = 0 \tag{2.39a}$$

$$N_x'' = 0 \tag{2.39b}$$

$$N_{xy}' = 0 \tag{2.39c}$$

$$N_{xy}'' = 0 \tag{2.39d}$$

$$({}_{o}N_x' + {}_{o}N_x'')\frac{\partial w}{\partial x} + \frac{\partial}{\partial x}(M_x' + M_x'') + 2\frac{\partial}{\partial y}(M_{xy}' + M_{xy}'')$$

$$+ \hat{t}\bar{Q}_x = 0 \tag{2.39e}$$

$$M_y' + M_y'' = 0 \tag{2.39f}$$

CHAPTER III

SOLUTION OF THE EQUATIONS

Transformation into an Uncoupled System

The solution of equations (2.20) is difficult because they are coupled in the four dependent variables u', u'', v', and v''. By a transformation of variables two sets of equations more amenable to solution may be developed. One set resulting from this transformation is composed of two equations in two variables. These equations are then the equations of plane elasticity including the effects of a body force. The second set of equations may be solved by a separation of variables.

To effect this transformation, new displacement components are defined as follows:

$$
\begin{aligned}
u &= \frac{m}{1+m}(u' - u'') \\
v &= \frac{m}{1+m}(v' - v'') \\
\tilde{u} &= \frac{1}{1+m}(u'' + mu') \\
\tilde{v} &= \frac{1}{1+m}(v'' + mv')
\end{aligned}
\tag{3.1}
$$

where

$$m = \frac{E't'}{E''t''}$$

It is possible to give a physical interpretation of the quantities in (3.1). The variation of $\hat{u}$ and the variation of $\hat{v}$ through the plate thickness are given by straight lines with discontinuities of the slope. Figure 2 illustrates this variation for $\hat{u}$. If, however, it is imagined that these variations are given by lines of constant slope, then for pure bending a neutral surface exists, which is defined by the quantities

$$h' = \frac{\bar{t}\hat{t}}{1 + m}$$

$$h'' = \frac{m}{1 + m}\,\bar{t}\hat{t}$$

where h' and h'' represent the distance from the neutral surface to the middle planes of the lower and upper faces respectively. Then $\tilde{u}$ and $\tilde{v}$ represent the displacements of the neutral surface, that is, since the neutral surface was defined for pure bending, $\tilde{u}$ and $\tilde{v}$ represent the displacements due purely to membrane effects. Furthermore, u and v represent the displacements of the middle surface of the upper face relative to the neutral surface of the plate.

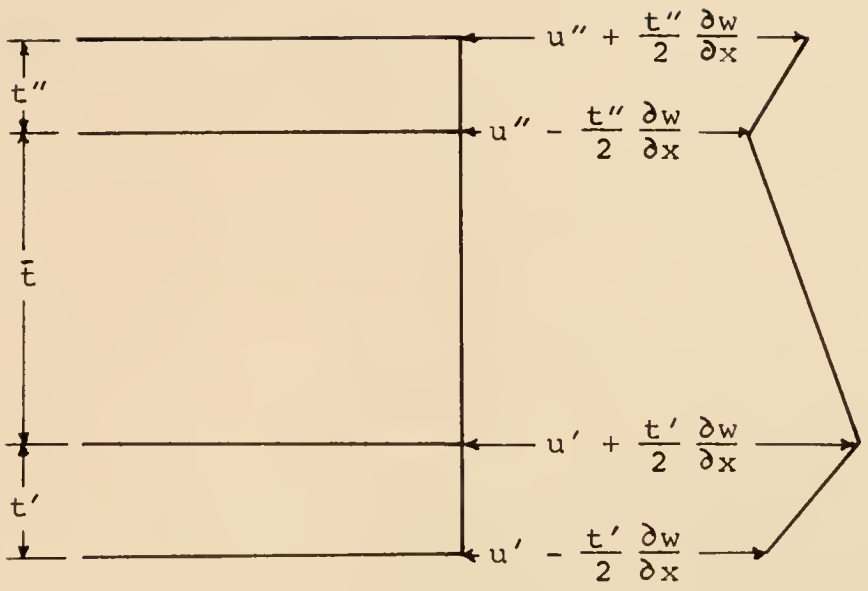

Figure 2. Variation of displacement û through the thickness of the sandwich panel

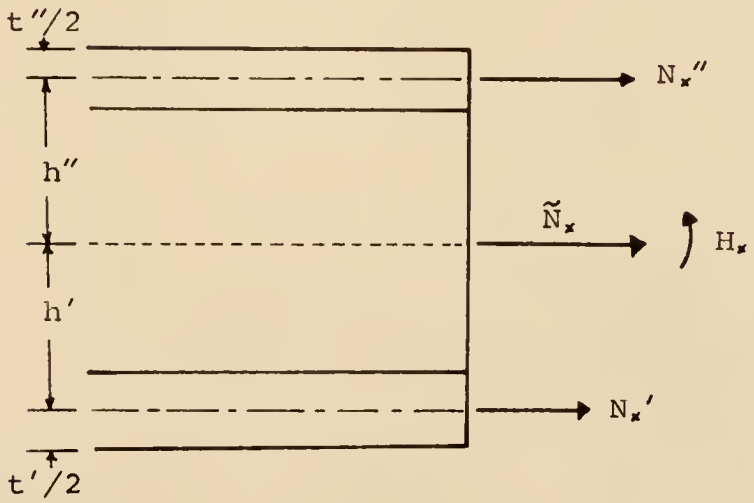

Figure 3. Resolution of membrane forces into a force and couple system at the "neutral" surface

Substitution of the transformations (3.1) into the equations for membrane force resultants (2.7) yields the corresponding transformations for these quantities:

$$
\begin{aligned}
{}_{o}\tilde{N}_{x} &= {}_{o}N_{x}' + {}_{o}N_{x}'' \\
{}_{o}\tilde{N}_{y} &= {}_{o}N_{y}' + {}_{o}N_{y}'' \\
{}_{o}\tilde{N}_{xy} &= {}_{o}N_{xy}' + {}_{o}N_{xy}''
\end{aligned}
$$

$$
\begin{aligned}
\tilde{N}_{x} &= N_{x}' + N_{x}'' \\
\tilde{N}_{y} &= N_{y}' + N_{y}'' \\
\tilde{N}_{xy} &= N_{xy}' + N_{xy}'' \\
\tilde{N}_{T} &= N_{T}' + N_{T}''
\end{aligned}
$$

$$
\begin{aligned}
H_{x} &= \frac{\bar{t}\hat{t}}{1 + m}(N_{x}' - mN_{x}'') \\
H_{y} &= \frac{\bar{t}\hat{t}}{1 + m}(N_{y}' - mN_{y}'') \\
H_{xy} &= \frac{\bar{t}\hat{t}}{1 + m}(N_{xy}' - mN_{xy}'') \\
H_{T} &= \frac{\bar{t}\hat{t}}{1 + m}(N_{T}' - mN_{T}'')
\end{aligned}
$$

where $\hat{t}$ has been defined in equation (2.14).

The quantities $\tilde{N}_{x}$ and H_{x} may be interpreted as the statically equivalent resolution of the membrane forces N_{x}' and N_{x}'', which act in the middle planes of the faces, into a force and couple system acting at the neutral surface (see

Fig. 3). The interpretation of the other quantities follows in a similar manner.

Substitution of the transformations (3.1) into the equations of equilibrium (2.20) yields

$$(D' + D'')\nabla^4 w - {}_o\tilde{N}_x \frac{\partial^2 w}{\partial x^2} - 2\,{}_o\tilde{N}_{xy} \frac{\partial^2 w}{\partial x \partial y} - {}_o\tilde{N}_y \frac{\partial^2 w}{\partial y^2} + \nabla^2 M_T$$

$$- \hat{t}\left[\bar{G}_x \left(\frac{1 + m}{m} \frac{\partial u}{\partial x} + \hat{t}\bar{t} \frac{\partial^2 w}{\partial x^2}\right) + \bar{G}_y \left(\frac{1 + m}{m} \frac{\partial v}{\partial y} + \hat{t}\bar{t} \frac{\partial^2 w}{\partial y^2}\right)\right]$$

$$- q'' = 0 \tag{3.2a}$$

$$\frac{E't'}{2m(1 - \mu^2)}\left[2 \frac{\partial^2 u}{\partial x^2} + (1 - \mu)\frac{\partial^2 u}{\partial y^2} + (1 + \mu)\frac{\partial^2 v}{\partial x \partial y}\right]$$

$$- \bar{G}_x \left(\frac{1 + m}{m\bar{t}} u + \hat{t} \frac{\partial w}{\partial x}\right) = \frac{1}{\bar{t}\hat{t}} \frac{\partial H_T}{\partial x} \tag{3.2b}$$

$$\frac{E't'}{2m(1 - \mu^2)}\left[2 \frac{\partial^2 v}{\partial y^2} + (1 - \mu)\frac{\partial^2 v}{\partial x^2} + (1 + \mu)\frac{\partial^2 u}{\partial x \partial y}\right]$$

$$- \bar{G}_y \left(\frac{1 + m}{m\bar{t}} v + \hat{t} \frac{\partial w}{\partial y}\right) = \frac{1}{\bar{t}\hat{t}} \frac{\partial H_T}{\partial y} \tag{3.2c}$$

$$\frac{\partial \tilde{N}_x}{\partial x} + \frac{\partial \tilde{N}_{xy}}{\partial y} = 0 \tag{3.3a}$$

$$\frac{\partial \tilde{N}_{xy}}{\partial x} + \frac{\partial \tilde{N}_y}{\partial y} = 0 \tag{3.3b}$$

Similarly, the boundary conditions become as follows.

(1) On the boundaries y = 0 and y = b

$$\tilde{N}_y = 0 \tag{3.4a}$$

$$\tilde{N}_{xy} = 0 \tag{3.4b}$$

$$u = \frac{m}{1 + m} \int_{a/2}^{x} (\alpha' T_2' - \alpha'' T_2'') \, dx \tag{3.5a}$$

$$H_y = 0 \tag{3.5b}$$

$$w = 0 \tag{3.5c}$$

$$M_y' + M_y'' = 0 \tag{3.5d}$$

(2) On the boundaries x = 0 and x = a, for the case of simply supported edges,

$$\tilde{N}_x = 0 \tag{3.6a}$$

$$\tilde{N}_{xy} = 0 \tag{3.6b}$$

$$H_x = 0 \tag{3.7a}$$

$$H_{xy} = 0 \tag{3.7b}$$

$$w = 0 \tag{3.7c}$$

$$M_y' + M_y'' = 0 \tag{3.7d}$$

For the case of clamped edges

$$\tilde{u} = 0 \tag{3.8a}$$

$$\tilde{N}_{xy} = 0 \tag{3.8b}$$

$$u = 0 \tag{3.9a}$$

$$H_{xy} = 0 \tag{3.9b}$$

$$w = 0 \tag{3.9c}$$

$$\frac{\partial w}{\partial x} = 0 \tag{3.9d}$$

For the case of free edges

$$\tilde{N}_x = 0$$

$$\tilde{N}_{xy} = 0$$

$$H_x = 0 \tag{3.10a}$$

$$H_{xy} = 0 \tag{3.10b}$$

$$M_y' + M_y'' = 0 \tag{3.10c}$$

$$\frac{\partial}{\partial x}(M_x' + M_x'') + 2\frac{\partial}{\partial y}(M_{xy}' + M_{xy}'') + \hat{t}\bar{Q}_x + ({}_oN_x' + {}_oN_x'')\frac{\partial w}{\partial x} = 0 \tag{3.10d}$$

Solution of the Membrane Force Equations

The membrane forces in the initial state of stress are governed by equations (1.2) and (1.3). In general, a complete solution for the initial state of stress requires the further specification of the compatibility condition. However, since the forces ${}_{o}N_{x}$, ${}_{o}N_{y}$, and ${}_{o}N_{xy}$ have been assumed to be an order of magnitude higher than the linear additional stresses, the strain-displacement relations will include nonlinear terms. As a result, the compatibility equation becomes nonlinear and, in general, a solution of the state of stress is not possible.

If the initial edge forces ${}_{o}N_{x}^{*}$, ${}_{o}N_{y}^{*}$, and ${}_{o}N_{xy}^{*}$ are constant, then the initial membrane forces are constant. Since ${}_{o}N_{x}^{*} = {}_{o}N_{xy}^{*} = 0$, there results

$$\begin{aligned} {}_{o}N_{x} &= {}_{o}N_{xy} = 0 \\ {}_{o}N_{y} &= {}_{o}N_{y}^{*} \end{aligned} \tag{3.11}$$

and therefore

$$\begin{aligned} {}_{o}\tilde{N}_{x} &= {}_{o}\tilde{N}_{xy} = 0 \\ {}_{o}\tilde{N}_{y} &= {}_{o}\tilde{N}_{y}^{*} \end{aligned} \tag{3.12}$$

In the case when the edges of the plate $x = 0$ and $x = a$ are clamped, the second set of boundary conditions (1.3) is replaced by the conditions

$$ {}_{o}u = 0 $$

$$ {}_{o}N_{xy} = 0 $$

The solution for the membrane forces requires the specification of the compatibility condition. If it is assumed that the plate is clamped after the initial stress is applied, then equations (3.12) will also apply to this case.

The membrane forces $\tilde{N}_x$, $\tilde{N}_y$, and $\tilde{N}_{xy}$ arising from the additional stresses satisfy equations (3.3) and boundary conditions (3.4) and (3.6). These equations are identical to those just discussed for the initial stresses. However, since these forces are within the realm of linear elasticity, the compatibility equation may be expressed as[27]

$$ \nabla^4(\tilde{N}_x + \tilde{N}_y) + (1 - \mu)\nabla^2\tilde{N}_T = 0 $$

Now the classical methods for plane stress problems[28] can be used for the solution of the state of stress.

If $N_x^* = N_y^* = N_{xy}^* = 0$ and N_T is a linear function of x and y, where

$$\tilde{N}_7 = \frac{1}{1-\mu}(E'\alpha' T_2' t' + E''\alpha'' T_2'' t'')$$

then the equilibrium equations, compatibility equation, and boundary conditions are satisfied by

$$\tilde{N}_x = \tilde{N}_y = \tilde{N}_{xy} = 0$$

and thus

$$\tilde{u} = \tilde{v} = 0$$

Solution of the Second Set of Equations

Once the membrane forces have been found as indicated in the preceding section, the next step is to solve the set of three partial differential equations (3.2), which are subject to the boundary conditions (3.5) and (3.7), (3.9), or (3.10).

First the system is transformed into dimensionless units. The scheme used by Ebcioglu[23] is followed. Dimensionless constants, variables, and reference quantities are defined as follows:

$$\xi = \frac{x}{a}$$

$$\eta = \frac{y}{b}$$

$$P_e = \frac{\pi^2}{b^2} \frac{E't'}{1 - \mu^2}(h')^2 (1 + m)$$

$$P_f = \frac{\pi^2}{b^2}(D' + D'')$$

$$k_{,e} = - \frac{{}_o\tilde{N}_y^*}{P_e}$$

$$r_e = \frac{P_e}{\hat{t}^2 \bar{t} \bar{G}_y}$$

$$\bar{g} = \frac{\bar{G}_x}{\bar{G}_y}$$

$$\beta = \frac{b}{a}$$

$$\rho = \frac{1 + m}{2} \hat{t}$$

$$\alpha = \frac{2\bar{t}}{b}$$

The temperature gradient is nondimensionalized by the introduction of the quantity

$$R = \frac{1 + \mu}{1 + m}(\alpha' T_2' - \alpha'' T_2'') \qquad (3.13)$$

The quantity P_e has been defined as a reference quantity with dimensions of force per unit length; it represents the buckling load of a simply supported, infinitely long sandwich strip with a rigid core, calculated under the

assumption that the faces act as membranes. P_f represents the sum of the buckling loads of each face of the strip.

Substitution of these quantities into the appropriate equations and boundary conditions yields the transformed equilibrium equations:

$$\frac{r_e}{\pi^2}\left[2\beta^2\frac{\partial^2 u}{\partial \xi^2} + (1-\mu)\frac{\partial^2 u}{\partial \eta^2} + \beta(1+\mu)\frac{\partial^2 v}{\partial \xi \partial \eta}\right] - 2\bar{g}u$$

$$- \frac{m\bar{g}\hat{t}^2\alpha\beta}{2\rho}\frac{\partial w}{\partial \xi} = \frac{2\beta m b r_e}{\pi^2}\frac{\partial R}{\partial \xi} \qquad (3.14a)$$

$$\frac{r_e}{\pi^2}\left[2\frac{\partial^2 v}{\partial \eta^2} + \beta^2(1-\mu)\frac{\partial^2 v}{\partial \xi^2} + \beta(1+\mu)\frac{\partial^2 u}{\partial \xi \partial \eta}\right] - 2v$$

$$- \frac{\hat{t}^2\alpha m}{2\rho}\frac{\partial w}{\partial \eta} = \frac{2mbr_e}{\pi^2}\frac{\partial R}{\partial \eta} \qquad (3.14b)$$

$$P_f\frac{m}{2\pi^2 b\bar{G}_y}\left[\beta^4\frac{\partial^4 w}{\partial \xi^4} + 2\beta^2\frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 w}{\partial \eta^4}\right] - \rho\bar{g}\beta\frac{\partial u}{\partial \xi} - \frac{\alpha\beta^2\hat{t}^2\bar{g}m}{4}\frac{\partial^2 w}{\partial \xi^2}$$

$$- \rho\frac{\partial v}{\partial \eta} + \frac{\alpha m\hat{t}^2(r_e k_{1e} - 1)}{4}\frac{\partial^2 w}{\partial \eta^2} = \frac{qmb}{2\bar{G}_y} - \frac{mb}{2\bar{G}_y}\nabla^2 M_T \qquad (3.14c)$$

and the transformed boundary conditions, written in terms of displacement components: (1) On the boundaries $\eta = 0$ and $\eta = 1$

$$u = \frac{mb}{\beta(1+\mu)} \int_{1/2}^{\xi} R(\xi)\, d\xi \tag{3.15a}$$

$$\frac{\partial v}{\partial \eta} + \beta\mu \frac{\partial u}{\partial \xi} = mbR \tag{3.15b}$$

$$w = 0 \tag{3.15c}$$

$$\frac{P_f}{\pi^2} \frac{\partial^2 w}{\partial \eta^2} = -M_T \tag{3.15d}$$

(2) On the boundaries $\xi = 0$ and $\xi = 1$, for the case of simply supported edges,

$$\beta \frac{\partial u}{\partial \xi} + \mu \frac{\partial v}{\partial \eta} = mbR \tag{3.16a}$$

$$\frac{\partial u}{\partial \eta} + \beta \frac{\partial v}{\partial \xi} = 0 \tag{3.16b}$$

$$w = 0 \tag{3.16c}$$

$$\beta^2 \frac{P_f}{\pi^2} \frac{\partial^2 w}{\partial \xi^2} = -M_T \tag{3.16d}$$

For the case of clamped edges

$$u = 0 \tag{3.17a}$$

$$\frac{\partial u}{\partial \eta} + \beta \frac{\partial v}{\partial \xi} = 0 \tag{3.17b}$$

$$w = 0 \tag{3.17c}$$

$$\frac{\partial w}{\partial \xi} = 0 \tag{3.17d}$$

For the case of free edges

$$\beta \frac{\partial u}{\partial \xi} + \mu \frac{\partial v}{\partial \eta} = mbR \tag{3.18a}$$

$$\frac{\partial u}{\partial \eta} + \beta \frac{\partial v}{\partial \xi} = 0 \tag{3.18b}$$

$$\frac{P_f}{\pi^2}\left(\beta^2 \frac{\partial^2 w}{\partial \xi^2} + \mu \frac{\partial^2 w}{\partial \eta^2}\right) = -M_T \tag{3.18c}$$

$$- \frac{P_f \beta}{\pi^2 b}\left[\beta^2 \frac{\partial^3 w}{\partial \xi^3} + (2 - \mu)\frac{\partial^3 w}{\partial \xi \partial \eta^2}\right] + \frac{1 + m}{m} \hat{t}\bar{G}_x u$$

$$+ \frac{\beta}{b}(\hat{t}^2 \bar{t}\bar{G}_x)\frac{\partial w}{\partial \xi} = \frac{\beta}{b} \frac{d}{d\xi} M_T \tag{3.18d}$$

It is seen that the system of equations (3.14) is linear and nonhomogeneous and is accompanied by nonhomogeneous boundary conditions. By the substitution of auxiliary functions[29] the nonhomogeneous boundary conditions on the sides $\eta = 0$ and $\eta = 1$ can be reduced to homogeneous ones. Then it is possible to select trigonometric functions of η that satisfy these homogeneous boundary conditions. These functions may be employed in the Levy method[25] of

separation of variables so that the partial differential equations are reduced to ordinary ones. It is necessary to expand the nonhomogeneous parts of the boundary conditions and equations in a series of these trigonometric functions; furthermore, the final solution is expressed in such series, and the convergence of these series is investigated in order to validate their differentiation.

In order to reduce the boundary conditions (3.15) to homogeneous form, the following substitutions are made:

$$u = u_o(\xi,\eta) + \eta\left[f_1(\xi) - f_o(\xi)\right] + f_o(\xi) \tag{3.19a}$$

$$v = v_o(\xi,\eta) + \eta^2\left[g_1(\xi) - \frac{1}{2}\beta\mu\frac{df_1(\xi)}{d\xi}\right] - (1-\eta)^2\left[g_o(\xi) - \frac{1}{2}\beta\mu\frac{df_o(\xi)}{d\xi}\right] \tag{3.19b}$$

$$w = w_o(\xi,\eta) + \frac{\eta(\eta-1)}{6}\left[(\eta+1)h_1(\xi) + (2-\eta)h_o(\xi)\right]\frac{\pi^2}{P_f} \tag{3.19c}$$

where the auxiliary functions denoted by f, g and h represent the following nonhomogeneous boundary terms:

$$f_o(\xi) = \frac{mb}{\beta(1+\mu)} \int_{1/2}^{\xi} R(\xi,0)\,d\xi$$

$$f_1(\xi) = \frac{mb}{\beta(1+\mu)} \int_{1/2}^{\xi} R(\xi,1)\,d\xi$$

$$g_o(\xi) = \frac{1}{2}\, mbR(\xi,0)$$

$$g_1(\xi) = \frac{1}{2}\, mbR(\xi,1)$$

$$h_o(\xi) = -M_T(\xi,0)$$

$$h_1(\xi) = -M_T(\xi,1)$$

It may be readily verified that the desired reduction is accomplished by means of substitutions (3.19). The equations governing the quantities u_o, v_o, and w_o are then established by the substitution of relations (3.19) into equations (3.14) through (3.18). The following equations result:

$$\frac{r_e}{\pi^2}\left[2\beta^2\frac{\partial^2 u_o}{\partial\xi^2} + (1-\mu)\frac{\partial^2 u_o}{\partial\eta^2} + \beta(1+\mu)\frac{\partial^2 v_o}{\partial\xi\,\partial\eta}\right] - 2\bar{g}u_o$$

$$- \frac{m\bar{g}\hat{t}^2\alpha\beta}{2\rho}\frac{\partial w_o}{\partial\xi} = \frac{2\beta mbr_e}{\pi^2}\frac{\partial R}{\partial\xi} + J_1(\xi,\eta) \qquad (3.20a)$$

$$\frac{r_e}{\pi^2}\left[2\frac{\partial^2 v_o}{\partial\eta^2} + \beta^2(1-\mu)\frac{\partial^2 v_o}{\partial\xi^2} + \beta(1+\mu)\frac{\partial^2 u_o}{\partial\xi\,\partial\eta}\right] - 2v_o$$

$$- \frac{\hat{t}^2\alpha m}{2\rho}\frac{\partial w_o}{\partial\eta} = \frac{2mbr_e}{\pi^2}\frac{\partial R}{\partial\eta} + J_2(\xi,\eta) \qquad (3.20b)$$

$$P_f\frac{m}{2\pi^2 b\bar{G}_y}\left[\beta^4\frac{\partial^4 w_o}{\partial\xi^4} + 2\beta^2\frac{\partial^4 w_o}{\partial\xi^2\partial\eta^2} + \frac{\partial^4 w_o}{\partial\eta^4}\right] - \rho\bar{g}\beta\frac{\partial u_o}{\partial\xi}$$

$$- \frac{\alpha\beta^2\hat{t}^2\bar{g}m}{4}\frac{\partial^2 w_o}{\partial\xi^2} - \rho\frac{\partial v_o}{\partial\eta} + \frac{\alpha m\hat{t}^2(r_e k_{1e} - 1)}{4}\frac{\partial^2 w_o}{\partial\eta^2}$$

$$= \frac{qmb}{2\bar{G}_y} - \frac{mb}{2\bar{G}_y}\nabla^2 M_T + J_3(\xi,\eta) \qquad (3.20c)$$

The boundary conditions on $\xi = 0$ and $\xi = 1$ become, for the case of simply supported edges,

$$\beta\frac{\partial u_o}{\partial\xi} + \mu\frac{\partial v_o}{\partial\eta} = mbR + S_1(\xi,\eta) \qquad (3.21a)$$

$$\frac{\partial u_o}{\partial\eta} + \beta\frac{\partial v_o}{\partial\xi} = S_2(\xi,\eta) \qquad (3.21b)$$

$$w_o = S_3(\xi,\eta) \qquad (3.21c)$$

$$\beta^2\frac{P_f}{\pi^2}\frac{\partial^2 w_o}{\partial\xi^2} = -M_T + S_4(\xi,\eta) \qquad (3.21d)$$

For the case of clamped edges

$$u_o = C_1(\xi,\eta) \qquad (3.22a)$$

$$\frac{\partial u_o}{\partial \eta} + \beta \frac{\partial v_o}{\partial \xi} = S_2(\xi, \eta) \tag{3.22b}$$

$$w_o = S_3(\xi, \eta) \tag{3.22c}$$

$$\frac{\partial w_o}{\partial \xi} = C_4(\xi, \eta) \tag{3.22d}$$

For the case of free edges

$$\beta \frac{\partial u_o}{\partial \xi} + \mu \frac{\partial v_o}{\partial \eta} = mbR + S_1(\xi, \eta) \tag{3.23a}$$

$$\frac{\partial u_o}{\partial \eta} + \beta \frac{\partial v_o}{\partial \xi} = S_2(\xi, \eta) \tag{3.23b}$$

$$\frac{P_f}{\pi^2}\left(\beta^2 \frac{\partial^2 w_o}{\partial \xi^2} + \mu \frac{\partial^2 w_o}{\partial \eta^2}\right) = -M_T + F_3(\xi, \eta) \tag{3.23c}$$

$$\frac{P_f \beta}{\pi^2 b}\left[\beta^2 \frac{\partial^3 w_o}{\partial \xi^3} + (2 - \mu)\frac{\partial^3 w_o}{\partial \xi \, \partial \eta^2}\right] - \frac{1 + m}{m} \hat{t}\bar{G}_x u_o - \frac{\beta}{b}(\hat{t}^2 \bar{t}\bar{G}_x)\frac{\partial w_o}{\partial \xi}$$

$$= -\frac{\beta}{b}\frac{dM_T}{d\xi} + F_4(\xi, \eta) \tag{3.23d}$$

In the above boundary conditions the functions $J_i(\xi, \eta)$, $S_i(\xi, \eta)$, $C_i(\xi, \eta)$, and $F_i(\xi, \eta)$, where i = 1, 2, 3, 4, represent the additional terms arising from the substitutions (3.19).

The boundary conditions on $\eta = 0$ and $\eta = 1$ are

$$u_o = 0 \tag{3.24a}$$

$$\frac{\partial v_o}{\partial \eta} + \beta\mu \frac{\partial u_o}{\partial \xi} = 0 \qquad (3.24b)$$

$$w_o = 0 \qquad (3.24c)$$

$$\frac{\partial^2 w_o}{\partial \eta^2} = 0 \qquad (3.24d)$$

It is seen that the boundary conditions (3.24) along the sides $\eta = 0$ and $\eta = 1$ are satisfied by terms of the form

$$u_{oj} = X_j(\xi) \sin j\pi\eta$$

$$v_{oj} = Y_j(\xi) \cos j\pi\eta$$

$$w_{oj} = Z_j(\xi) \sin j\pi\eta$$

Thus the following solutions are assumed:

$$u_o = \sum_{k=1}^{\infty} X_j(\xi) \sin j\pi\eta \qquad (3.25a)$$

$$v_o = \sum_{k=1}^{\infty} Y_j(\xi) \cos j\pi\eta \qquad (3.25b)$$

$$w_o = \sum_{k=1}^{\infty} Z_j(\xi) \sin j\pi\eta \qquad (3.25c)$$

However, before attempting an infinite series approach, certain mathematical questions must be examined.

Convergence of Series Solutions

Once solutions of the form of (3.25) have been proposed, it is desirable to develop some assurance that the series can satisfy the required equations, in other words, that term-by-term differentiation of the series will produce a series converging to a proper derivative.

Such considerations must not be thought a mere mathematical superfluity. Quantities of physical interest occurring in the deformation of a plate involve second and third derivatives of the displacements; if the displacements are represented as infinite series, it may happen that term-by-term differentiation does not provide series that converge to their physical counterparts. For example, the deflection y of a vibrating beam of length π, simply supported, with an applied sinusoidal bending moment at one end is represented by the following equation:[30]

$$\frac{\partial^2 y}{\partial t^2} + a^2 \frac{\partial^4 y}{\partial x^4} = 0$$

with the boundary conditions

$$y(0,t) = y(\pi,t) = 0$$

$$\frac{\partial^2 y}{\partial x^2}(0,t) = 0$$

$$\frac{\partial^2 y}{\partial x^2}(\pi,t) = b \sin \omega t$$

If a solution is attempted by the method of finite sine transforms, the result will be the infinite series

$$y = \frac{2}{\pi} \sum \frac{a^2 bn(-1)^{n+1}}{a^2 n^4 - \omega^2} \left(\sin \omega t - \frac{\omega}{an^2} \sin an^2 t\right) \sin nx$$

which, upon term-by-term differentiation, will be found to satisfy neither the differential equation nor the nonhomogeneous boundary condition; nor will the internal bending moments calculated in this way converge to their true values.

In order to be assured that term-by-term differentiation of a convergent series, particularly a Fourier series, will lead to another convergent series, two theorems of pertinence may be noted. The first is as follows.[31]

> Theorem 1. Let there be given a series $\sum_{n=0}^{\infty} f_n(x)$ whose terms are differentiable in the interval $J = [a,b]$ and which converges at least at one point of J. If the series $\sum_{n=0}^{\infty} f_n'(x)$ deduced from it by term-by-term differentiation converges uniformly in J, then so does the given series. Furthermore, if $\sum_{n=0}^{\infty} f_n(x) = F(x)$ and $\sum_{n=0}^{\infty} f_n'(x) = \varphi(x)$, then
>
> $$F'(x) = \varphi(x)$$

It is therefore necessary first to perform formally the

term-by-term differentiation and then to investigate the uniform convergence of the result in order to determine the validity of the operation.

The second theorem relates specifically to the term-by-term differentiation of Fourier series.[32]

> Theorem 2. If $f(x)$ is a continuous function of period 2π with an absolutely integrable derivative, then the Fourier series of $f'(x)$ can be obtained from the Fourier series of $f(x)$ by term-by-term differentiation.

A corollary of this theorem applicable to the interval $(0,1)$ is given as follows.[32]

> Theorem 2(a). If $f(x)$ is continuous and absolutely integrable on $(0,1)$, then the Fourier cosine series $\sum a_n \cos n\pi x$ of $f(x)$ on this interval may be differentiated term by term, while the Fourier sine series $\sum b_n \sin n\pi x$ can be differentiated term by term if $f(0) = f(1) = 0$.

It is seen that the Fourier cosine series is necessarily always continuous everywhere; if the conditions in the corollary are met, the sine series is also continuous everywhere.

Theorem 2 applies to functions of one variable only. However, it may be extended to series of the form of equations (3.25) if the variable ξ is treated as a parameter.

It is now possible to proceed with the justification of the assumed solutions. From Theorem 2(a) and boundary condition (3.24a) the series for u_o may be twice differentiated with respect to η. Condition (3.24a) implies that $\frac{\partial u_o}{\partial \xi} = 0$ along $\eta = 0$ and $\eta = 1$, and thus condition (3.24b) may be replaced by

$$\frac{\partial v_o}{\partial \eta} = 0 \tag{3.26}$$

The cosine series (3.25b) for v_o can be differentiated once with respect to η, and condition (3.26) implies the permissibility of a second differentiation. Finally, conditions (3.24c) and (3.24d), together with Theorem 2(a), imply that the series for w_o may be differentiated four times with respect to η.

It is concluded that it is permissible to perform all partial differentiations with respect to η required to satisfy the differential equations and boundary conditions.

The validity of taking derivatives with respect to ξ, according to Theorem 1, can be established only after the actual solution is obtained and the differentiations in

question performed. Some further remarks will be made when this stage in the analysis is reached.

Development and Solution of the Ordinary Differential Equations

The next step in the solution involves the development of ordinary differential equations in the variable ξ. These relations arise from equating coefficients of trigonometric terms after substituting the assumed solutions (3.25) into equations (3.20). In this process the nonhomogeneous terms of equations (3.20) are expanded in Fourier sine or cosine series on the interval (0,1) according to the following formulas.

(1) The Fourier cosine expansion of a function f(x) defined on $0 \leq x \leq 1$ is

$$f(x) = \frac{a_o}{2} + \sum_{j=1}^{\infty} a_j \cos j\pi x$$

where

$$a_j = 2 \int_0^1 f(x) \cos j\pi x dx$$

(2) The Fourier sine expansion of a function f(x) defined on $0 \leq x \leq 1$ is

$$f(x) = \sum_{j=1}^{\infty} b_j \sin j\pi x$$

where

$$b_j = 2\int_0^1 f(x) \sin j\pi x dx$$

In this way the following relations may be written:

$$\frac{\partial R}{\partial \xi} = \sum_{j=1}^{\infty} K_{1j}(\xi) \sin j\pi\eta$$

$$\frac{\partial R}{\partial \eta} = \sum_{j=1}^{\infty} K_{2j}(\xi) \cos j\pi\eta + \frac{1}{2} K_{20}(\xi)$$

$$q = \sum_{j=1}^{\infty} K_{3j}(\xi) \sin j\pi\eta$$

$$\nabla^2 M_T = \sum_{j=1}^{\infty} K_{4j}(\xi) \sin j\pi\eta$$

Furthermore, the remaining terms on the right-hand sides of equations (3.20) are given by

$$J_1(\xi,\eta) = \sum_{j=1}^{\infty} H_{1j}(\xi) \sin j\pi\eta$$

$$J_2(\xi,\eta) = \sum_{j=1}^{\infty} H_{2j}(\xi) \cos j\pi\eta + \frac{1}{2} H_{20}(\xi)$$

$$J_3(\xi,\eta) = \sum_{j=1}^{\infty} H_{3j}(\xi) \sin j\pi\eta$$

In a similar manner the nonhomogeneous right-hand sides of the boundary conditions (3.21) through (3.23) on $\xi = 0$ and $\xi = 1$ are

$$mbR(\xi,\eta) + S_1(\xi,\eta) = \sum_{j=1}^{\infty} B_{aj}(\xi) \sin j\pi\eta \qquad (3.27a)$$

$$S_2(\xi,\eta) = \sum_{j=1}^{\infty} B_{cj}(\xi) \cos j\pi\eta + \frac{1}{2} B_{c0}(\xi) \qquad (3.27b)$$

$$S_3(\xi,\eta) = \sum_{j=1}^{\infty} B_{ej}(\xi) \sin j\pi\eta \qquad (3.27c)$$

$$-M_T(\xi,\eta) + S_4(\xi,\eta) = \sum_{j=1}^{\infty} B_{gj}(\xi) \sin j\pi\eta \qquad (3.27d)$$

$$C_1(\xi,\eta) = \sum_{j=1}^{\infty} C_{aj}(\xi) \sin j\pi\eta \qquad (3.27e)$$

$$C_4(\xi,\eta) = \sum_{j=1}^{\infty} C_{gj}(\xi) \sin j\pi\eta \qquad (3.27f)$$

$$-M_T(\xi,\eta) + F_3(\xi,\eta) = \sum_{j=1}^{\infty} F_{ej}(\xi) \sin j\pi\eta \qquad (3.27g)$$

$$-\frac{\beta}{b}\frac{dM_T(\xi,\eta)}{d\xi} + F_4(\xi,\eta) = \sum_{j=1}^{\infty} F_{gj}(\xi) \sin j\pi\eta \qquad (3.27h)$$

As a result of applying the procedure described above, the following systems of ordinary differential equations result (D denotes the operator $\frac{d}{d\xi}$):

$$\left\{\frac{2r_e\beta^2}{\pi^2} D^2 - \left[r_e(1-\mu)j^2 + 2\bar{g}\right]\right\} X_j - \frac{r_e\beta(1+\mu)j}{\pi} DY_j$$

$$- \frac{m\bar{g}\hat{t}^2\alpha\beta}{2\rho} DZ_j = \frac{2\beta mbr_e}{\pi^2} K_{ij} + H_{ij} \qquad (3.28a)$$

$$\frac{r_e\beta(1+\mu)j}{\pi} DX_j + \left[\frac{r_e\beta^2(1-\mu)}{\pi^2} D^2 - (2r_e j^2 + 2)\right] Y_i$$

$$- \frac{\hat{t}^2\alpha mj\pi}{2\rho} Z_j = \frac{2mbr_e}{\pi^2} K_{2j} + H_{2j} \qquad (3.28b)$$

$$-\rho\bar{g}\beta DX_j + \rho j\pi Y_j + \left\{\frac{P_f m\beta^4}{2\pi^2 b\bar{G}_y} D^4 - \left[\frac{P_f m\beta^2 j^2}{b\bar{G}_y} + \frac{\alpha\beta^2\hat{t}^2\bar{g}m}{4}\right] D^2\right.$$

$$\left. + \left[\frac{P_f mj^4\pi^2}{2b\bar{G}_y} + \frac{\alpha m\hat{t}^2(1-r_e k_{1e})j^2\pi^2}{4}\right]\right\} Z_j$$

$$= \frac{mb}{2\bar{G}_y} K_{3j} + \frac{mb}{2\bar{G}_y} K_{4j} + H_{3j} \qquad (3.28c)$$

$$\left[\frac{r_e}{2\pi^2}\beta^2(1-\mu)D^2 - 1\right]Y_o = \frac{mbr_e}{\pi^2}K_{2o} + \frac{1}{2}H_{2o} \tag{3.28d}$$

where $j = 1, 2, \ldots$.

The boundary conditions (3.21) on $\xi = 0$ and $\xi = 1$ become, for the case of simply supported edges,

$$\beta\frac{dX_j(\xi)}{d\xi} - \mu j\pi Y_j(\xi) = 2B_{oj}(\xi) \tag{3.29a}$$

$$j\pi X_j(\xi) + \beta\frac{dY_j}{d\xi} = B_{cj}(\xi) \tag{3.29b}$$

$$Z_j(\xi) = B_{ej}(\xi) \tag{3.29c}$$

$$\frac{d^2Z_j(\xi)}{d\xi} = B_{gj}(\xi) \tag{3.29d}$$

$$\beta\frac{dY_o(\xi)}{d\xi} = B_{co}(\xi) \tag{3.29e}$$

where $j = 1, 2, \ldots$.

The boundary conditions for the clamped and free edges similarly follow.

The complementary and particular parts of the solutions to equations (3.28) are denoted by the subscripts c and p respectively as follows:

$$X_j = X_{cj} + X_{pj} \tag{3.30a}$$

$$Y_j = Y_{cj} + Y_{pj} \tag{3.30b}$$

$$Z_j = Z_{cj} + Z_{pj} \tag{3.30c}$$

$$Y_o = Y_{oc} + Y_{op} \tag{3.30d}$$

Since equations (3.28) are linear with constant coefficients, solutions of the following form are assumed:

$$X_{cj} = A_j e^{\lambda_j \xi} \tag{3.31a}$$

$$Y_{cj} = B_j e^{\lambda_j \xi} \tag{3.31b}$$

$$Z_{cj} = C_j e^{\lambda_j \xi} \tag{3.31c}$$

$$Y_{oc} = B_o e^{\lambda_o \xi} \tag{3.31d}$$

Substitution of the above relations into the homogeneous form of equations (3.28) reveals that λ_j satisfies the following quartic equation:

$$\left(r_e\theta_j - \frac{2}{1-\mu}\right)\left[\frac{2P_f}{b\bar{G}_y\hat{t}^2\alpha}\theta_j^3 - \bar{g}\left(1 + \frac{2P_f}{b\bar{G}_y r_e\hat{t}^2\alpha}\right)\theta_j^2 - r_e k_{1e} j^2\theta_j + \bar{g}j^2 k_{1e}\right] + j^2 r_e(\bar{g} - 1)\left\{\left[\frac{2P_f}{b\bar{G}_y r_e\hat{t}^2\alpha}\left(\frac{2}{1-\mu} - 1\right) - 1\right]\theta_j^2 + j^2 k_{1e}\left(1 - \frac{2}{1-\mu}\right)\right\} = 0 \tag{3.32}$$

where

$$\Theta_j = \left(\frac{\beta \lambda_j}{\pi}\right)^2 - j^2$$

Further,

$$\lambda_o = \pm \frac{\pi}{\beta} \sqrt{\frac{2}{r_e(1 - \mu)}}$$

If there are no repeated roots of equation (3.32), the complementary solutions are given by

$$X_{cj} = \sum_{i=1}^{8} A_{ji} e^{\lambda_{ji} \xi} \tag{3.33a}$$

$$Y_{cj} = \sum_{i=1}^{8} B_{ji} e^{\lambda_{ji} \xi} \tag{3.33b}$$

$$Z_{cj} = \sum_{i=1}^{8} C_{ji} e^{\lambda_{ji} \xi} \tag{3.33c}$$

$$Y_{oc} = B_{o1} e^{\lambda_o \xi} + B_{11} e^{-\lambda_o \xi} \tag{3.33d}$$

The constants B_{ji} and C_{ji} are related to the constants A_{ji} by the relations

$$B_{ji} = b_{ji} A_{ji}$$

$$C_{ji} = c_{ji} A_{ji}$$

where

$$b_{ji} = \frac{j\pi}{\beta\lambda_{ji}} \left\{ \frac{r_e\beta^2\left[2 - \bar{g}(1+\mu)\right]\lambda_{ji}^2 - \pi^2\left[r_e(1-\mu)j^2 + 2\bar{g}\right]}{r_e\bar{g}\beta^2(1-\mu)\lambda_{ji}^2 - \pi^2\left[2\bar{g}(r_e j^2 + 1) - j^2 r_e(1+\mu)\right]} \right\} \tag{3.34}$$

$$c_{ji} = \frac{2\rho}{m\beta\bar{g}\hat{t}^2\alpha\lambda_{ji}} \left\{ \frac{2r_e\beta^2\lambda_{ji}^2}{\pi^2} - (1-\mu)j^2 r_e - 2\bar{g} - \frac{r_e(1+\mu)j\beta\lambda_{ji}}{\pi} b_{ji} \right\}$$

The constants A_{ji} are determined by substituting the complete solutions (3.30) into boundary conditions (3.29); eight simultaneous algebraic equations result, and their solution may be written in the form

$$A_{ji} = \frac{D_{ji}}{\Delta_j} \tag{3.35}$$

where Δ_j is an eight-by-eight determinant

$$\Delta_j = \left| a_{\kappa L} \right|$$

whose elements are given as follows for the case of simply supported edges:

$$a_{1L} = (\beta\lambda_{jL} - \mu j\pi b_{jL})e^{\lambda_{jL}} \qquad a_{1,L+1} = -(\beta\lambda_{jL} - \mu j\pi b_{jL})e^{-\lambda_{jL}}$$

$$a_{2L} = (\beta\lambda_{jL} - \mu j\pi b_{jL}) \qquad a_{2,L+1} = -(\beta\lambda_{jL} - \mu j\pi b_{jL})$$

$$a_{3L} = (j\pi + \beta\lambda_{jL} b_{jL})e^{\lambda_{jL}} \qquad a_{3,L+1} = (j\pi + \beta\lambda_{jL} b_{jL})e^{-\lambda_{jL}}$$

$$a_{4L} = j\pi + \beta\lambda_{jL} b_{jL} \qquad a_{4,L+1} = j\pi + \beta\lambda_{jL} b_{jL}$$

$$a_{5L} = c_{jL} e^{\lambda_{jL}} \qquad a_{5,L+1} = -c_{jL} e^{-\lambda_{jL}}$$

$$a_{6L} = c_{jL} \qquad a_{6,L+1} = -c_{jL}$$

$$a_{7L} = \lambda_{jL}^2 c_{jL} e^{\lambda_{jL}} \qquad a_{7,L+1} = -\lambda_{jL}^2 c_{jL} e^{-\lambda_{jL}}$$

$$a_{8L} = \lambda_{jL}^2 c_{jL} \qquad a_{8,L+1} = -\lambda_{jL}^2 c_{jL}$$

where L = 1, 3, 5, 7. The D_{ji} are the determinants formed by replacing the i-th column of Δ_j by the column vector

$$\left\| \begin{array}{l} 2B_{oj}(1) - \beta \dfrac{dX_{pj}(1)}{d\xi} + \mu j\pi Y_{pj}(1) \\ 2B_{oj}(0) - \beta \dfrac{dX_{pj}(0)}{d\xi} + \mu j\pi Y_{pj}(0) \\ B_{cj}(1) - j\pi X_{pj}(1) - \beta \dfrac{dY_{pj}(1)}{d\xi} \\ B_{cj}(0) - j\pi X_{pj}(0) - \beta \dfrac{dY_{pj}(0)}{d\xi} \\ B_{ej}(1) - Z_{pj}(1) \\ B_{ej}(0) - Z_{pj}(0) \\ B_{gj}(1) - \dfrac{d^2 Z_{pj}(1)}{d\xi^2} \\ B_{gj}(0) - \dfrac{d^2 Z_{pj}(0)}{d\xi^2} \end{array} \right\|$$

Also,

$$B_{o1} = \frac{e^{-\lambda_o}\left[B_{co}(0) - \beta \dfrac{dY_{op}(0)}{d\xi}\right] - B_{co}(1) + \beta \dfrac{dY_{op}(1)}{d\xi}}{\beta\lambda_o(e^{-\lambda_o} - e^{\lambda_o})}$$

$$B_{11} = \frac{e^{\lambda_o}\left[B_{co}(0) - \beta \dfrac{dY_{op}(0)}{d\xi}\right] - B_{co}(1) + \beta \dfrac{dY_{op}(1)}{d\xi}}{\beta\lambda_o(e^{-\lambda_o} - e^{\lambda_o})}$$

For the case when the edges $\xi = 0$ and $\xi = 1$ are clamped, the elements of Δ_j are given by

$$a_{1L} = e^{\lambda_{jL}} \qquad a_{1,L+1} = e^{-\lambda_{jL}}$$

$$a_{2L} = 1 \qquad a_{2,L+1} = 1$$

$$a_{3L} = (j\pi + \beta\lambda_{jL} b_{jL}) e^{\lambda_{jL}} \qquad a_{3,L+1} = (j\pi + \beta\lambda_{jL} b_{jL}) e^{-\lambda_{jL}}$$

$$a_{4L} = j\pi + \beta\lambda_{jL} b_{jL} \qquad a_{4,L+1} = j\pi + \beta\lambda_{jL} b_{jL}$$

$$a_{5L} = c_{jL} e^{\lambda_{jL}} \qquad a_{5,L+1} = -c_{jL} e^{-\lambda_{jL}}$$

$$a_{6L} = c_{jL} \qquad a_{6,L+1} = -c_{jL}$$

$$a_{7L} = \lambda_{jL} c_{jL} e^{\lambda_{jL}} \qquad a_{7,L+1} = \lambda_{jL} c_{jL} e^{-\lambda_{jL}}$$

$$a_{8L} = \lambda_{jL} c_{jL} \qquad a_{8,L+1} = \lambda_{jL} c_{jL}$$

where L = 1, 3, 5, 7. The D_{ji} are formed from the column vector

$$\left\| \begin{array}{l} C_{aj}(1) - X_{pj}(1) \\ C_{aj}(0) - X_{pj}(0) \\ B_{cj}(1) - j\pi X_{pj}(1) - \beta \dfrac{dY_{pj}(1)}{d\xi} \\ B_{cj}(0) - j\pi X_{pj}(0) - \beta \dfrac{dY_{pj}(0)}{d\xi} \\ B_{ej}(1) - Z_{pj}(1) \\ B_{ej}(0) - Z_{pj}(0) \\ C_{gj}(1) - \dfrac{dZ_{pj}(1)}{d\xi} \\ C_{gj}(0) - \dfrac{dZ_{pj}(0)}{d\xi} \end{array} \right\|$$

For the case when the edges $\xi = 0$ and $\xi = 1$ are free, there results

$$a_{1L} = (\beta\lambda_{jL} - \mu j\pi b_{jL})e^{\lambda_{jL}}$$

$$a_{2L} = \beta\lambda_{jL} - \mu j\pi b_{jL}$$

$$a_{3L} = (j\pi + \beta\lambda_{jL} b_{jL})e^{\lambda_{jL}}$$

$$a_{4L} = j\pi + \beta\lambda_{jL} b_{jL}$$

$$a_{5L} = (\beta^2\lambda_{jL}^2 c_{jL} - \mu j^2\pi^2 c_{jL})e^{\lambda_{jL}}$$

$$a_{6L} = \beta^2\lambda_{jL} c_{jL} - \mu j^2\pi^2 c_{jL}$$

$$a_{7L} = \left\{\frac{P_f\beta}{\pi^2 b}\left[\beta^2\lambda_{jL}^3 - (2-\mu)j^2\pi^2\lambda_{jL}\right]c_{jL} - \frac{1+m}{m}\hat{t}\bar{G}_x - \frac{\beta}{b}(\hat{t}^2\bar{t}\bar{G}_x)c_{jL}\lambda_{jL}\right\}e^{\lambda_{jL}}$$

$$a_{8L} = \frac{P_f\beta}{\pi^2 b}\left[\beta^2\lambda_{jL}^3 - (2-\mu)j^2\pi^2\lambda_{jL}\right]c_{jL} - \frac{1+m}{m}\hat{t}\bar{G}_x - \frac{\beta}{b}(\hat{t}^2\bar{t}\bar{G}_x)c_{jL}\lambda_{jL}$$

$$a_{1,L+1} = -(\beta\lambda_{jL} - \mu j\pi b_{jL})e^{-\lambda_{jL}}$$

$$a_{2,L+1} = -(\beta\lambda_{jL} - \mu j\pi b_{jL})$$

$$a_{3,L+1} = (j\pi + \beta\lambda_{jL} b_{jL})e^{-\lambda_{jL}}$$

$$a_{4,L+1} = j\pi + \beta\lambda_{jL} b_{jL}$$

$$a_{5,L+1} = -(\beta^2\lambda_{jL}^2 c_{jL} - \mu j^2\pi^2 c_{jL})e^{-\lambda_{jL}}$$

$$a_{6,L+1} = -(\beta^2\lambda_{jL}^2 c_{jL} - \mu j^2\pi^2 c_{jL})$$

$$a_{7,L+1} = \left\{\frac{P_f\beta}{\pi^2 b}\left[\beta^2\lambda_{jL}^3 - (2-\mu)j^2\pi^2\lambda_{jL}\right]c_{jL} - \frac{1+m}{m}\hat{t}\bar{G}_x - \frac{\beta}{b}(\hat{t}^2\bar{t}\bar{G}_x)c_{jL}\lambda_{jL}\right\}e^{-\lambda_{jL}}$$

$$a_{g,L+1} = \frac{P_f \beta}{\pi^2 b}\left[\beta^2 \lambda_{jL}^3 - (2-\mu) j^2 \pi^2 \lambda_{jL}\right] c_{jL} - \frac{1+m}{m}\hat{t}\bar{G}_x$$

$$- \frac{\beta}{b}(\hat{t}^2 \bar{t}\bar{G}_x) c_{jL} \lambda_{jL}$$

where L = 1, 3, 5, 7. The D_{jL} are formed from the column vector

$$\left\| \begin{array}{l}
2B_{aj}(1) - \beta \dfrac{dX_{pj}(1)}{d\xi} + \mu j \pi Y_{pj}(1) \\
2B_{aj}(0) - \beta \dfrac{dX_{pj}(0)}{d\xi} + \mu j \pi Y_{pj}(0) \\
B_{cj}(1) - j\pi X_{pj}(1) - \beta \dfrac{dY_{pj}(1)}{d\xi} \\
B_{cj}(0) - j\pi X_{pj}(0) - \beta \dfrac{dY_{pj}(0)}{d\xi} \\
\dfrac{\pi^2}{P_f} F_{ej}(1) - \beta^2 \dfrac{d^2 Z_{pj}(1)}{d\xi^2} + \mu j^2 \pi^2 Z_{pj}(1) \\
\dfrac{\pi^2}{P_f} F_{ej}(0) - \beta^2 \dfrac{d^2 Z_{pj}(0)}{d\xi^2} + \mu j^2 \pi^2 Z_{pj}(0) \\
F_{gj}(1) - \dfrac{P_f \beta}{\pi^2 b}\left[\beta^2 \dfrac{d^3 Z_{pj}(1)}{d\xi^3} - (2-\mu) j^2 \pi^2 \dfrac{dZ(1)}{d\xi}\right] \\
\qquad + \dfrac{1+m}{m}\hat{t}\bar{G}_x X_{pj}(1) + \dfrac{\beta}{b}(\hat{t}^2 \bar{t}\bar{G}_x)\dfrac{dZ_{pj}(1)}{d\xi} \\
F_{gj}(0) - \dfrac{P_f \beta}{\pi^2 b}\left[\beta^2 \dfrac{d^3 Z_{pj}(0)}{d\xi^3} - (2-\mu) j^2 \pi^2 \dfrac{dZ(0)}{d\xi}\right] \\
\qquad + \dfrac{1+m}{m}\hat{t}\bar{G}_x X_{pj}(0) + \dfrac{\beta}{b}(\hat{t}^2 \bar{t}\bar{G}_x)\dfrac{dZ_{pj}(0)}{d\xi}
\end{array} \right\|$$

Therefore, from equations (3.19), (3.25), (3.30), (3.31), and (3.35), the solutions for the displacement

components u, v, and w are

$$u(\xi,\eta) = \eta f_1(\xi) + (1-\eta) f_0(\xi) + \sum_{j=1}^{\infty} \left\{ \sum_{i=1}^{8} \frac{D_{ji}}{\Delta_j} e^{\lambda_{ji}\xi} + X_{pj}(\xi) \right\} \sin j\pi\eta \qquad (3.36a)$$

$$v(\xi,\eta) = \eta^2 \left[g_1(\xi) - \frac{\beta\mu}{2} \frac{df_1(\xi)}{d\xi} \right] - (1-\eta)^2 \left[g_0(\xi) - \frac{\beta\mu}{2} \frac{df_0(\xi)}{d\xi} \right] + \sum_{j=1}^{\infty} \left\{ \sum_{i=1}^{8} b_{ji} \frac{D_{ji}}{\Delta_j} e^{\lambda_{ji}\xi} + Y_{pj}(\xi) \right\} \cos j\pi\eta + \frac{1}{2} \left[B_{01} e^{\lambda_0\xi} + B_{11} e^{-\lambda_0\xi} + Y_{0p}(\xi) \right] \qquad (3.36b)$$

$$w(\xi,\eta) = \frac{\pi^2}{6P_f} \eta(\eta-1) \left[(\eta+1)h_1(\xi) + (2-\eta)h_0(\xi) \right] + \sum_{j=1}^{\infty} \left\{ \sum_{i=1}^{8} c_{ji} \frac{D_{ji}}{\Delta_j} e^{\lambda_{ji}\xi} + Z_{pj}(\xi) \right\} \sin j\pi\eta \qquad (3.36c)$$

These equations completely determine the displacement components.

Now that the solutions have been completely specified, further remarks can be made about the validity of forming partial derivatives of these quantities with respect to ξ. Unfortunately, because of the extreme complexity of the terms in each infinite series, it is not possible to

actually examine the uniform convergence of the derivatives. All that can be said is that actual evaluation of the series for w at the center of the plate indicated that the terms converged. Uniform convergence was not shown.

Solution for a Simplified Case

In the preceding part of this analysis the solution of a very general case of loading on a sandwich plate has been presented. The remainder of the analysis will utilize certain restrictions which aid in the tractability of the problem, while still representing a broad category of results.

The following assumptions are now made:

1. Bending of the faces is neglected ($P_f = 0$);
2. The temperature is constant through the thickness of each face;
3. The temperature distribution and the edge loading are symmetric with respect to perpendicular bisectors of the edges of the plate;
4. The transverse loading q" is constant.

As a result of these restrictions, the following relations hold:

$$f_1(\xi) = f_o(\xi) = \frac{mb}{\beta(1+\mu)} \int_{1/2}^{\xi} R(\xi,1)\,d\xi \qquad (3.37a)$$

$$g_1(\xi) = g_o(\xi) = mbR \tag{3.37b}$$

$$h_1(\xi) = h_o(\xi) = 0 \tag{3.37c}$$

$$M_T = 0 \tag{3.37d}$$

The particular solutions can be determined from equations (3.28) after the following relations are noted:

$$T_1(\xi,\eta) = -\frac{r_e \beta mb}{\pi^2}\left(\frac{2}{1+\mu} - \mu + 1 + \mu\right)\frac{\partial R}{\partial \xi}(\xi,1)$$

$$+ 2\bar{g}\,\frac{mb}{\beta(1+\mu)}\int_{1/2}^{\xi} R(\xi,1)\,d\xi \tag{3.38a}$$

$$T_2(\xi,\eta) = -\frac{r_e \beta^2}{2\pi^2}(1-\mu)(2\eta-1)mb\left(1 - \frac{\mu}{1+\mu}\right)\frac{\partial^2 R}{\partial \xi^2}$$

$$+ (2\eta-1)\left[mb\left(1 - \frac{\mu}{1+\mu}\right)R(\xi,1)\right] \tag{3.38b}$$

$$T_3(\xi,\eta) = \rho\left[mb\left(1 + \bar{g}\,\frac{1}{1+\mu} - \frac{\mu}{1+\mu}\right)R(\xi,1)\right] \tag{3.38c}$$

and after the following series expansions are noted:

$$1 = \frac{4}{\pi}\sum_{j=1,3\ldots}^{\infty}\frac{1}{j}\sin j\pi\eta \tag{3.39a}$$

$$\eta = \frac{1}{2} - \frac{4}{\pi^2} \sum_{j=1,3\ldots}^{\infty} \frac{1}{j^2} \cos j\pi\eta \tag{3.39b}$$

$$\frac{\partial R(\xi,\eta)}{\partial \xi} = 2 \sum_{j=1,3\ldots}^{\infty} \int_0^1 \frac{\partial R(\xi,\eta)}{\partial \xi} \sin j\pi\eta \, d\eta \sin j\pi\eta \tag{3.39c}$$

$$\frac{\partial R(\xi,\eta)}{\partial \eta} = 2 \sum_{j=1,3\ldots}^{\infty} \int_0^1 \frac{\partial R(\xi,\eta)}{\partial \eta} \cos j\pi\eta \, d\eta \cos j\pi\eta \tag{3.39d}$$

where the series in the last two equations contain only terms with an odd-numbered index because of the symmetry of $R(\xi,\eta)$. Thus from equations (3.38) and (3.39) there result

$$K_{1j} = 2 \int_0^1 \frac{\partial R}{\partial \xi} \sin j\pi\eta \, d\eta \tag{3.40a}$$

$$K_{2j} = 2 \int_0^1 \frac{\partial R}{\partial \eta} \cos j\pi\eta \, d\eta \tag{3.40b}$$

$$K_{3j} = \frac{4}{\pi j} q_o \tag{3.40c}$$

$$K_{4j} = 0 \tag{3.40d}$$

$$K_{20} = 0 \tag{3.40e}$$

$$H_{1j} = -\frac{4}{\pi}\left[\frac{r_e\beta}{\pi^2}\,mb\left(\frac{2}{1+\mu}-\mu+1+\mu\right)\frac{\partial R(\xi,1)}{\partial \xi^2}\right.$$

$$\left. -\,2\bar{g}\,\frac{mb}{\beta(1+\mu)}\int_{1/2}^{\xi} R(\xi,1)\,d\xi\right]\frac{1}{j} \qquad (3.40f)$$

$$H_{2j} = -\frac{8}{\pi^2}\left\{\frac{r_e}{2\pi^2}\,\beta^2(\mu-1)\,mb\left(1-\frac{\mu}{1+\mu}\right)\frac{\partial^2 R(\xi,1)}{\partial \xi^2}\right.$$

$$\left. +\left[mb\left(1-\frac{\mu}{1+\mu}\right)R(\xi,1)\right]\right\}\frac{1}{j^2} \qquad (3.40g)$$

$$H_{3j} = \frac{4}{\pi}\rho\left[mb\left(1+\bar{g}\,\frac{1}{1+\mu}-\frac{\mu}{1+\mu}\right)R(\xi,1)\right]\frac{1}{j} \qquad (3.40h)$$

$$H_{20} = 0 \qquad (3.40i)$$

where $j = 1, 3, 5, \ldots$.

Then the particular solutions X_{pj}, Y_{pj}, and Z_{pj} are found from equations (3.28) after P_f has been set equal to zero and expressions (3.40) have been substituted for the quantities on the right-hand sides of these equations. Furthermore, it is found that for any symmetrical temperature distribution

$$Y_{0p} = 0$$

The particular solutions for two specific temperature distributions were found. For a constant distribution $R = R_o$ these solutions are

$$X_{pj} = A_j \xi + B_j$$

$$Y_{pj} = C_j \tag{3.41}$$

$$Z_{pj} = D_j$$

where

$$A_j = - \frac{8mb\bar{g}}{\pi\beta(1 + \mu) j \left[r_e(1 - \mu) j^2 + 2\bar{g}\right]} R_o$$

$$B_j = - \frac{1}{2} A_j$$

$$C_j = \frac{\gamma_j \zeta_j - \beta_j \kappa_j}{\alpha_j \zeta_j - \beta_j \delta_j}$$

$$D_j = \frac{\alpha_j \kappa_j - \gamma_j \delta_j}{\alpha_j \zeta_j - \beta_j \delta_j}$$

and

$$\alpha_j = 2(r_e j^2 + 1)$$

$$\beta_j = \frac{\hat{t}^2 \alpha m j \pi}{2\rho}$$

$$\gamma_j = \frac{8}{\pi^2 j^2}\left[mb\left(1 - \frac{\mu}{1+\mu}\right)R_o\right] - \frac{8r_e mb\bar{g}R_o}{\pi^2\left[r_e(1-\mu)j^2 + 2\bar{g}\right]}$$

$$\delta_j = \rho j\pi$$

$$\zeta_j = \frac{\alpha m\hat{t}^2(1 - r_e k_{1e})\pi^2 j^2}{4}$$

$$\kappa_j = \frac{4}{\pi j}\left\{\frac{mbq_o}{2\bar{G}_y} + \rho\left[mb\left(1 + \frac{\bar{g} - \mu}{1+\mu}\right)R_o\right]\right\}$$

$$- \frac{8\rho mb\bar{g}^2 R}{\pi j(1+\mu)\left[r_e(1-\mu)j^2 + 2\bar{g}\right]}$$

Here $j = 1, 3, 5, \ldots$.

For a double sinusoidal distribution of the form $R = R_1 \sin \pi\xi \sin \pi\eta$, the solutions for $j = 1$ are

$$X_{pj} = A_j \cos \pi\xi$$

$$Y_{pj} = E_j \sin \pi\xi + F_j \qquad (3.42)$$

$$Z_{pj} = H_j \sin \pi\xi + I_j$$

where

$$A_j = \varepsilon_j \frac{P_{1j}}{P_j}$$

$$E_j = \varepsilon_j \frac{P_{2j}}{P_j}$$

$$H_j = \varepsilon_j \frac{P_{3j}}{P_j}$$

Here

$$\Phi_j = -\frac{2mbr_e}{\pi} R_1$$

P_j is the determinant

$$\begin{vmatrix} \nu_j & \chi_j & \varphi_j \\ \chi_j & \Psi_j & \beta_j \\ \Gamma_j & \rho & \omega_j \end{vmatrix}$$

where

$$\nu_j = 2r_e\beta^2 + r_e(1-\mu) + 2\bar{g}$$

$$\chi_j = r_e\beta(1+\mu)$$

$$\Gamma_j = \rho\bar{g}\beta$$

$$\Psi_j = r_e\beta^2(1-\mu) + 2r_e + 2$$

$$\varphi_j = \frac{m\bar{g}\hat{t}^2\alpha\beta}{2\rho}\pi$$

$$\omega_j = \frac{\pi\alpha\hat{t}^2 m}{4}(\beta^2\bar{g} - r_e k_{1e} + 1)$$

and P_{ij} is the determinant formed by replacing the i-th column of P_j by the column vector

$$\begin{Vmatrix} \beta \\ 1 \\ 0 \end{Vmatrix}$$

Further,

$$F_j = -\Lambda_j \frac{\beta_j}{\alpha_j \zeta_j - \beta_j \delta_j}$$

$$I_j = \Lambda_j \frac{\alpha_j}{\alpha_j \zeta_j - \beta_j \delta_j}$$

where

$$\Lambda_j = \frac{4}{\pi j} \frac{mbq_c}{2\bar{G}_\gamma}$$

For $j = 3, 5, 7, \ldots$

$$A_j = E_j = H_j = 0$$

The coefficients in (3.27) are determined from equations (3.37) as

$$B_{oj}(0) = B_{oj}(1) = \frac{4}{\pi}\left\{mb\left[\mu + (1-\mu)\right]R(1,1)\right\}\frac{1}{j} + 2mb\int_0^1 R(1,\eta)\sin j\pi\eta\, d\eta \qquad (3.43a)$$

$$B_{cj}(1) = -B_{cj}(0) = \frac{4}{\pi^2}\beta mb\frac{\partial R(1,1)}{\partial \xi}\left(1 - \frac{\mu}{1+\mu}\right)\frac{1}{j^2} \qquad (3.43b)$$

$$B_{ej}(1) = B_{ej}(0) = 0 \qquad (3.43c)$$

$$B_{gj}(1) = B_{gj}(0) = 0 \qquad (3.43d)$$

$$B_{co}(1) = B_{co}(0) = 0 \qquad (3.43e)$$

$$C_{aj}(1) = C_{aj}(0) = 0 \tag{3.43f}$$

$$C_{gj}(1) = C_{gj}(0) = 0 \tag{3.43g}$$

$$F_{ej}(1) = F_{ej}(0) = 0 \tag{3.43h}$$

$$F_{gj}(1) = -F_{gj}(0) = \left[\frac{4(1+m)\hat{t}G_x b}{\pi\beta(1+\mu)} \int_0^1 R(\xi,1)\,d\xi\right] \frac{1}{j} \tag{3.43i}$$

where $j = 1, 3, 5, \ldots$.

The assumption of a temperature distribution symmetric in the ξ-direction makes possible a simplification of the complementary solutions (3.33). The function $E_j(\xi)$ is defined as

$$E_j(\xi) = e^{\lambda_j(\xi - 1/2)} + e^{-\lambda_j(\xi - 1/2)}$$

and is seen to be even with respect to the line $\xi = 1/2$, which bisects the plate; the function

$$O_j(\xi) = e^{\lambda_j(\xi - 1/2)} - e^{-\lambda_j(\xi - 1/2)}$$

is seen to be odd with respect to $\xi = 1/2$. Each pair of terms of the form

$$A_{ji}\, e^{\lambda_{ji}\xi} + A_{j,i+1}\, e^{-\lambda_j \xi}$$

that occurs in the complementary solutions (3.33) may be

written in terms of the newly defined functions as

$$\frac{1}{2}(A_{j1}\, e^{\frac{1}{2}\lambda_j} + A_{j,i+1}\, e^{-\frac{1}{2}\lambda_j})E_j + \frac{1}{2}(A_{j1}\, e^{\frac{1}{2}\lambda_j} - A_{j,i+1}\, e^{-\frac{1}{2}\lambda_j})O$$

Under the present assumptions, the horizontal displacement component u will be odd with respect to $\xi = 1/2$, while the horizontal displacement component v and the vertical displacement component w will be even. The expression (3.36a) for u in terms of the functions E_j and O_j becomes

$$u = \frac{mb}{\beta(1+\mu)} \int_{1/2}^{\xi} R(\xi,1)\,d\xi + \sum_{j=1,3}^{\infty} \left\{ \sum_{k=1}^{3} \frac{1}{2} \left[\left(\frac{D_{j,2k-1}}{\Delta_j}\, e^{\frac{1}{2}\lambda_{jk}} \right.\right.\right.$$

$$\left. + \frac{D_{j,2k}}{\Delta_j}\, e^{-\frac{1}{2}\lambda_{jk}} \right) E_{jk} + \left(\frac{D_{j,2k-1}}{\Delta_j}\, e^{\frac{1}{2}\lambda_{jk}} \right.$$

$$\left.\left.\left. - \frac{D_{j,2k}}{\Delta_j}\, e^{-\frac{1}{2}\lambda_{jk}} \right) O_{jk} \right] \right\} \sin j\pi\eta$$

Since u must be odd and the functions E_{jk} and O_{jk} are linearly independent, the coefficients of E_{jk} must equal zero; there result

$$D_{j2} = -e^{\lambda_{j1}}\, D_{j1}$$

$$D_{j4} = -e^{\lambda_{j2}}\, D_{j3}$$

$$D_{j6} = -e^{\lambda_{j3}}\, D_{j5}$$

Therefore the solutions (3.36) become

$$u = \frac{mb}{\beta(1+\mu)} \int_{1/2}^{\xi} R(\xi,1)\,d\xi + \sum_{j=1,3,\ldots}^{\infty} \left\{ \sum_{i=1}^{3} \left[e^{\lambda_{ji}\xi} - e^{\lambda_{ji}(1-\xi)} \right] \frac{D_{ji}}{\Delta_j} + X_{pj} \right\} \sin j\pi\eta \tag{3.44a}$$

$$v = \frac{2\eta - 1}{2} \left[mb\left(1 - \frac{\mu}{1+\mu}\right) R(\xi,1) \right] + \sum_{j=1,3,\ldots}^{\infty} \left\{ \sum_{i=1}^{3} b_{ji} \frac{D_{ji}}{\Delta_j} \left[e^{\lambda_{ji}\xi} + e^{\lambda_{ji}(1-\xi)} \right] + Y_{pj} \right\} \cos j\pi\eta \tag{3.44b}$$

$$w = \sum_{j=1,3,\ldots}^{\infty} \left\{ \sum_{i=1}^{3} c_{ji} \frac{D_{ji}}{\Delta_j} \left[e^{\lambda_{ji}\xi} + e^{\lambda_{ji}(1-\xi)} \right] + Z_{pj} \right\} \sin j\pi\eta \tag{3.44c}$$

Here the λ_{ji} are the roots of the cubic equation

$$\left(r_e\theta_j - \frac{2}{1-\mu}\right)\left(\bar{g}\theta_j^2 + r_e k_{1e} j^2\theta_j - \bar{g} j^2 k_{1e}\right) + j^2 r_e(\bar{g} - 1)\left[\theta_j^2 - j^2 k_{1e}\left(1 - \frac{2}{1-\mu}\right)\right] = 0 \tag{3.45}$$

where

$$\theta_j = \left(\frac{\beta\lambda_j}{\pi}\right)^2 - j^2$$

Further, Δ_j is the three-by-three determinant

$$\Delta_j = \left| a_{KL} \right| \tag{3.46}$$

where K,L = 1, 2, 3; the elements a_{KL} are, for the case when the edges $\xi = 0$ and $\xi = 1$ are simply supported,

$$a_{1L} = (e^{\lambda_{jL}} + 1)(\beta\lambda_{jL} - \mu j\pi b_{jL})$$

$$a_{2L} = (e^{\lambda_{jL}} - 1)(j\pi + \beta\lambda_{jL} b_{jL})$$

$$a_{3L} = (e^{\lambda_{jL}} + 1)c_{jL}$$

where L = 1, 2, 3. The D_{ji} are the determinants formed by replacing the i-th column of Δ_j by the column vector

$$\left\| \begin{array}{l} 2B_{oj}(1) - \beta \dfrac{dX_{pj}(1)}{d\xi} + \mu j\pi Y_{pj}(1) \\ \\ B_{cj}(1) - j\pi X_{pj}(1) - \beta \dfrac{dY_{pj}(1)}{d\xi} \\ \\ -Z_{pj}(1) \end{array} \right\| \tag{3.47}$$

where the coefficients B_{oj} and B_{cj} are given by expressions (3.43a) and (3.43b).

For the case when the edges $\xi = 0$ and $\xi = 1$ are clamped, the elements of Δ_j are

$$a_{1L} = e^{\lambda_{jL}} - 1$$

$$a_{2L} = (e^{\lambda_{jL}} - 1)(j\pi + \beta\lambda_{jL} b_{jL})$$

$$a_{3\ell} = (e^{\lambda_{j\ell}} + 1)c_{j\ell}$$

and the D_{ji} are formed from the column vector

$$\begin{Vmatrix} -X_{pj}(1) \\ B_{cj}(1) - j\pi X_{pj}(1) - \beta \dfrac{dY_{pj}(1)}{d\xi} \\ -Z_{pj}(1) \end{Vmatrix}$$

For the case when the edges $\xi = 0$ and $\xi = 1$ are free, the elements of Δ_j are

$$a_{1\ell} = (e^{\lambda_{j\ell}} + 1)(\beta\lambda_{j\ell} - \mu j\pi b_{j\ell})$$

$$a_{2\ell} = (e^{\lambda_{j\ell}} - 1)(j\pi + \beta\lambda_{j\ell} b_{j\ell})$$

$$a_{3\ell} = (e^{\lambda_{j\ell}} - 1)\left[-\frac{1+m}{m}\hat{t}\bar{G}_x - \frac{\beta}{b}(\hat{t}^2\bar{t}\bar{G}_x)c_{j\ell}\lambda_{j\ell}\right]$$

and the D_{ji} are formed from the column vector

$$\begin{Vmatrix} 2B_{aj}(1) - \beta \dfrac{dX_{pj}(1)}{d\xi} + \mu j\pi Y_{pj}(1) \\ B_{cj}(1) - j\pi X_{pj}(1) - \beta \dfrac{dY_{pj}(1)}{d\xi} \\ F_{gj}(1) - \dfrac{1+m}{m}\hat{t}\bar{G}_x X_{pj}(1) - \dfrac{\beta}{b}(\hat{t}^2\bar{t}\bar{G}_x) \dfrac{dZ_{pj}(1)}{d\xi} \end{Vmatrix}$$

The coefficients F_{gj} are given by expression (3.43i). The quantities b_{ji} and c_{ji} are given by expressions (3.34)

An observation can be made on the roots of the characteristic equation (3.45). If the equation is expanded into a cubic equation of standard form

$$\Lambda^3 + a\Lambda^2 + b\Lambda + c = 0 \tag{3.48}$$

where

$$\Lambda = \lambda_j^2$$

it is seen from the definitions and physical meaning of the coefficients that

$$a < 0$$

$$b > 0$$

$$c < 0 \quad \text{for} \quad k_{xe} < \frac{j^2}{1 + r_e j^2}$$

$$c > 0 \quad \text{for} \quad k_{xe} > \frac{j^2}{1 + r_e j^2}$$

From Descartes' rule of signs it follows that the number of positive real roots of the cubic equation will be either one or three if $k_{xe} < \frac{j^2}{1 + r_e j^2}$ or zero or two if $k_{xe} > \frac{j^2}{1 + r_e j^2}$. In general, then, as k_{xe}, which represents the compressive edge loading, increases past this value, complex values of λ_j are to be expected.

A further observation is that when ${}_oN_{yy} = 0$, i.e., no compressive edge loading is present, a double root for Λ_j occurs; in this case the solutions (3.44) approach the following limits as ${}_oN_{yy} \to 0$:

$$u = \frac{mb}{\beta(1+\mu)} \int_{1/2}^{\xi} R(\xi,1)\,d\xi + \sum_{j=1,3}^{\infty} \left\{ \frac{D_{j3}}{\Delta_j} \left[\xi e^{\lambda_{j1}\xi} - (1-\xi) e^{\lambda_{j1}(1-\xi)} \right] \right.$$

$$\left. + \sum_{i=1}^{2} \frac{D_{ji}}{\Delta_j} \left[e^{\lambda_{ji}\xi} - e^{\lambda_{ji}(1-\xi)} \right] + X_{pj} \right\} \sin j\pi\eta \qquad (3.49a)$$

$$v = (2\eta - 1)\frac{mb}{2}\left(1 - \frac{\mu}{1+\mu}\right) R(\xi,1)$$

$$+ \sum_{j=1,3,\ldots}^{\infty} \left[\frac{D_{j3}}{\Delta_j} \left\{ b_{j1} \left[\xi e^{\lambda_{j1}\xi} + (1-\xi) e^{\lambda_{j1}(1-\xi)} \right] \right.\right.$$

$$\left. + e_{j1} \left[e^{\lambda_{j1}\xi} + e^{\lambda_{j1}(1-\xi)} \right] \right\}$$

$$\left. + \sum_{i=1}^{2} b_{ji} \left[e^{\lambda_{ji}\xi} + e^{\lambda_{ji}(1-\xi)} \right] \frac{D_{ji}}{\Delta_j} + Y_{pj} \right] \cos j\pi\eta \qquad (3.49b)$$

$$w = \sum_{j=1,3,\ldots}^{\infty} \left[\left\{ c_{j1} \left[\xi e^{\lambda_{j1}} + (1-\xi) e^{\lambda_{j1}(1-\xi)} \right] \right.\right.$$

$$+ g_{j1}\left[e^{\lambda_{j1}\xi} + e^{\lambda_{j1}(1-\xi)}\right]\Bigg\}\frac{D_{j3}}{\Delta_j}$$

$$+ \sum_{i=1}^{2} c_{ji}\left[e^{\lambda_{ji}\xi} + e^{\lambda_{ji}(1-\xi)}\right]\frac{D_{ji}}{\Delta_j} + Z_{pj}\Bigg]\sin j\pi\eta \qquad (3.49c)$$

Here the λ_{ji} are the roots of the equation and are easily found to be

$$\lambda_{j1} = \frac{j\pi}{\beta}$$

$$\lambda_{j2} = \frac{\pi}{\beta}\sqrt{\frac{2\bar{g} + r_e j^2(1-\mu)}{r_e\bar{g}(1-\mu)}}$$

and Δ_j represents the determinant

$$\Delta_j = |a_{KL}|$$

whose elements are given by

$$a_{13} = \beta\left[1 + (1+\lambda_{j1})e^{\lambda_{j1}}\right] - \mu j\pi\left[e_{j1} + (b_{j1} + e_{j1})e^{\lambda_{j1}}\right]$$

$$a_{23} = j\pi e^{\lambda_{j1}} + \beta\left\{-(\lambda_{j1}e_{j1} + b_{j1}) + \left[(b_{j1} + e_{j1})\lambda_{j1} + b_{j1}\right]e^{\lambda_{j1}}\right\}$$

$$a_{33} = g_{j1} + (c_{j1} + g_{j1})e^{\lambda_{j1}}$$

$$a_{1L} = (1 + e^{\lambda_{jL}})(\beta\lambda_{jL} - \mu j\pi b_{jL})$$

$$a_{2L} = (e^{\lambda_{jL}} - 1)(j\pi + \beta b_{jL}\lambda_{jL})$$

$$a_{3L} = (e^{\lambda_{jL}} + 1)c_{jL}$$

where L = 1, 2. The D_{ji} are the determinants formed by replacing the i-th column of Δ_j by the column vector (3.47). The b_{ji} and c_{ji} are given by equations (3.34);

$$e_{ji} = \frac{\beta}{\pi j} \frac{r_e j^2 (3 - \mu)(1 - \bar{g}) + 2\bar{g}}{r_e j^2 (1 + \mu)(1 - \bar{g}) - 2\bar{g}}$$

$$g_{ji} = - \frac{4\rho\beta}{m\hat{t}^2 \alpha \pi^2 j^2} \left\{ r_e j^2 (3 - \mu) + \bar{g}\left[r_e j^2 (1 + \mu) + 2 \right] \right\}$$

while X_{pj}, Y_{pj}, and Z_{pj} are the particular solutions of equations (3.28).

CHAPTER IV

COMPARISON WITH EXPERIMENTAL RESULTS

Although there are many reports of experiments on rectangular sandwich panels without thermal effects, for example, those of Boller[33] and Kommers and Norris,[34] the only experiment involving thermal deformation known to the author is the one performed by Ebcioglu et al.[24]

In this experiment a rectangular sandwich panel was simply supported on all four edges and a temperature gradient was applied by simultaneously heating and cooling opposite faces of the panel. No compressive load was applied and the deflection was measured at the midpoint of the panel. The properties of the panel were given as follows:

$$
\begin{aligned}
& t' = t'' = 0.010 \text{ in.} \\
& \bar{t} = 0.395 \text{ in.} \\
& \mu = 0.3 \\
& E' = E'' = 30 \times 10^6 \text{ psi} \\
& \bar{G}_x = 39{,}825 \text{ psi} \\
& \bar{G}_y = 66{,}400 \text{ psi} \\
& a = b = 6 \text{ in.}
\end{aligned}
\tag{4.1}
$$

and thus

$$k_{/e} = 0$$

$$P_e = 1{,}853 \text{ lb/in}$$

$$\hat{t} = 1.0253$$

$$r_e = 0.067214$$

$$\alpha' = \alpha'' = 7.34 \times 10^{-6} \text{ in/in } {}^{o}F \qquad (4.2)$$

$$q_o = 0$$

$$\bar{g} = 0.6$$

$$m = 1$$

$$\beta = 1$$

From Ebcioglu et al. and the original data sheets of the experiment, it was found that for the four trials made in the experiment the distribution of the temperature gradient R, defined by equation (3.13), could be approximated by multiples of the function

$$R = (1.25 \sin \pi\eta + 1.373 \sin \pi\eta \sin \pi\xi) 10^{-4} \qquad (4.3)$$

A measure of the error caused by representing the temperature distribution by equation (4.3) was calculated from the formula

$$\text{Error} = \frac{\sum |a_i - e_i|}{\Delta T_{max} \sum i} \qquad (4.4)$$

where the a_i are the experimentally recorded values of the temperature gradient, the c_i are the values of the gradient calculated at the same points from formula (4.3), and ΔT_{max} is the maximum difference among the experimentally recorded values of the temperature gradient.

The following table gives the temperature gradient distribution, experimentally recorded values of the deflection at the center of the plate, and measure of the error as given by equation (4.4) for each trial.

TABLE 1

EXPERIMENTAL DATA

Trial	Temperature Gradient	Error	Measured Deflection (in.)
1	R	+0.005 -0.056	0.013
2	1.1 R	+0.023 -0.089	0.0135
3	1.2 R	+0.043 -0.079	0.0195
4	1.8 R	+0.065 -0.080	0.021

Since no edge loading was applied, the analytical solution to this problem is given by equation (3.49c). The particular solutions for the temperature distribution R as given in equation (4.3) are

$$X_p = -(0.262 \cos \pi\xi)10^{-3}$$

$$Y_p = -(0.477 + 0.262 \sin \pi\xi)10^{-3} \qquad (4.5)$$

$$Z_p = (9.011 + 5.066 \sin \pi\xi)10^{-3}$$

for $j = 1$, and

$$X_{pj} = 0$$

$$Y_{pj} = 0$$

$$Z_{pj} = 0$$

for $j = 3, 5, \ldots$.

Substitution of the values given in (4.1) and (4.2) and the particular solutions specified above into equation (3.49c) produces the theoretical results given in Table 2 for the four trials reported in the experiment. The comparison between the theoretical and experimental results is illustrated in Figure 4.

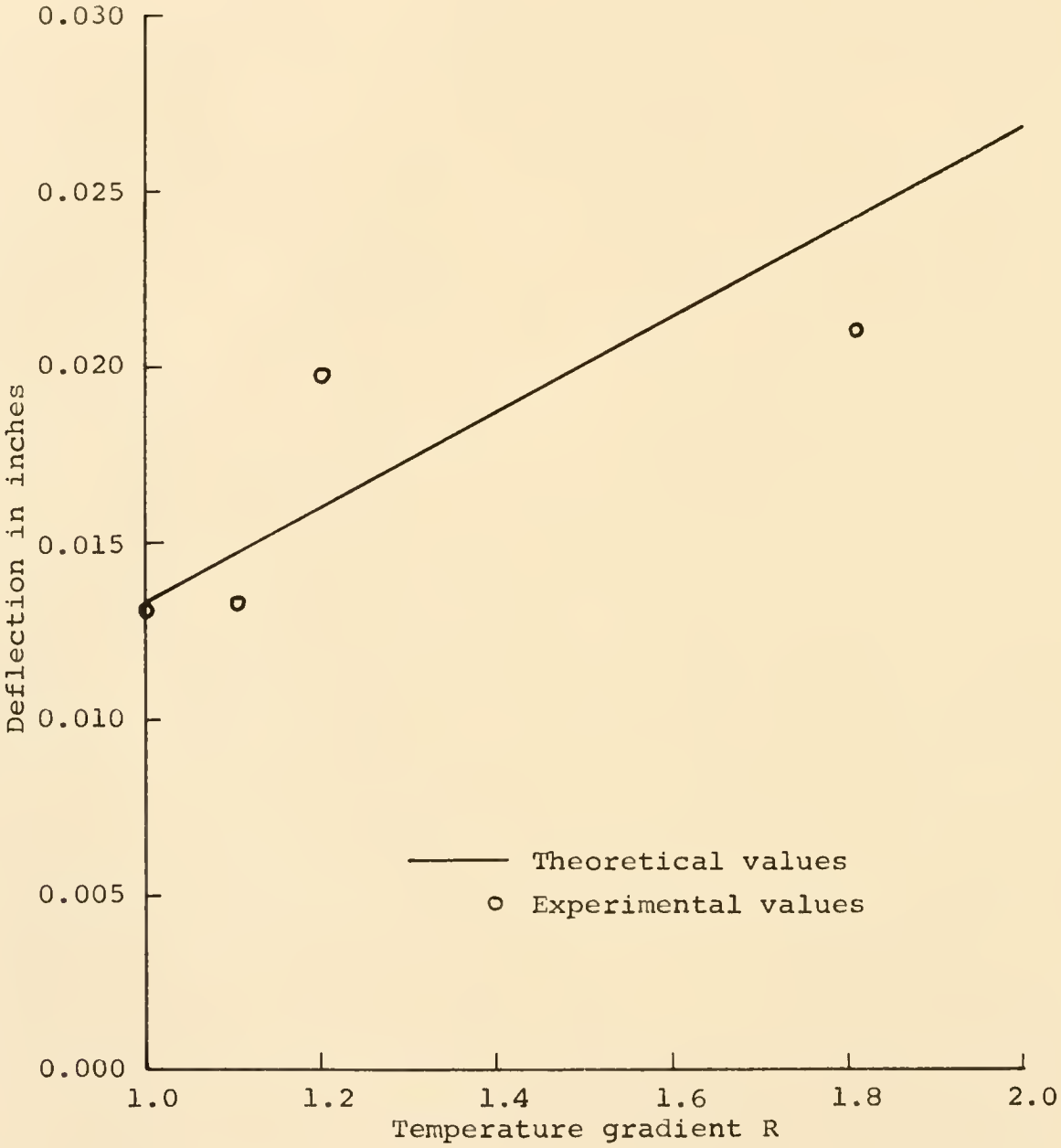

Figure 4. Comparison of theoretical and experimental values of the deflection at the center of the plate

TABLE 2

THEORETICAL RESULTS

Trial	Theoretically Calculated Deflection (in.)	% Error
1	0.01335	0.0
2	0.01468	8.9
3	0.01602	17.9
4	0.02403	14.3

The agreement with the experiment is seen to be excellent for trial 1, good for trial 2, and poor for the last two trials. Nevertheless, the results, while somewhat inconclusive, do tend to indicate satisfactory agreement with the theory, since, as Table 1 shows, the assumed temperature distribution most closely fits the actual distribution in trial 1. Furthermore, the experimental data, as presented in the appendix, show that for all four cases there is some doubt whether the experimental temperature distribution was actually symmetrical in the y-direction; since data were taken at only two points in the half of the plate necessary to establish this fact, no definite conclusion can be

reached. For trials 2, 3, and 4 the data show a definite lack of symmetry in the ξ-direction. It is felt, therefore, that the data of trial 1 are the most significant, and the large errors for the other three trials are not alarming. The need for further experimental evidence is indicated.

CHAPTER V

COMPUTATION AND DISCUSSION OF THE SOLUTION

Computation of the Solution for Various Parameters of the Panel

In order to investigate the dependence of the deflection and buckling of the panel on its properties and dimensions, equation (3.44c) was evaluated on the University of Florida IBM 709 computer for a variety of cases. It was decided to limit consideration to panels with faces of equal thickness, which are simply supported on all four edges and are not subject to a transverse load; in this way the number of variables was reduced to manageable size, while the ability was retained to vary such quantities as the ratio of face thickness to core thickness, aspect ratio of the plate, shear and Young's moduli, and the temperature distribution and edge compression.

The properties and dimensions of sandwich panels used in practice are as varied as their applications. Reference to industrial and design journals produced values of the dimensions ranging from $t' = t'' = 0.006$ in. and $\bar{t} = 0.44$ in.,

as used in the wing of a supersonic bomber,[35] to $t' = t'' = 0.064$ in. and $\bar{t} = 18$ in., as used in curtain wall construction. Honeycomb core shear moduli varied from $\bar{G}_x = 9{,}600$ psi and $\bar{G}_y = 21{,}600$ psi to $\bar{G}_x = 145{,}000$ psi and $\bar{G}_y = 81{,}000$ psi.[36] The usual facing materials were steel, with a Young's modulus $E = 30 \times 10^6$ psi, and aluminum, with $E = 10 \times 10^6$ psi.

A useful range of variation of the dimensionless parameter r_e, given by the relation

$$r_e = \frac{\pi^2 E' t' \bar{t}}{2b^2(1 - \mu)\bar{G}_y}$$

can be chosen from the above information as

$$0.01 < r_e < 2.00$$

The values $r_e = 0.017$ and $\bar{g} = 0.6$ given in the experiments reported by Ebcioglu[24] were chosen as typical values. A typical value of the aspect ratio β was selected as 2.0.

The nondimensionalization of the deflection w was accomplished by the definition of $\hat{w}$ as

$$\hat{w} = \frac{w}{R_o b/2\hat{t}\alpha}$$

where R_o is the maximum value over the surface of the plate of the temperature gradient function R given in equation

(3.13). The quantity $\hat{w}$ was calculated as a function of the applied edge loading parameter k_{le}. The calculations were performed for the two temperature distributions

$$R = R_o = \text{constant}$$

$$R = R_o \sin \pi\xi \sin \pi\eta$$

for varying r_e and for varying β.

The effect on the deflection of the sandwich panel due to the degree of lengthwise thermal expansion of the insert was also examined. Boundary condition (3.15a) was derived under the assumption that the thermal expansion of the edges $\eta = 0$ and $\eta = 1$ was not inhibited by the insert. At the opposite extreme is the assumption that no thermal expansion at all is allowed. The deflection was calculated for both extremes for varying r_e.

A computer program, written in the FORTRAN symbolic language, was designed in order to accomplish the tasks delineated above. This program instructed the computer to evaluate equation (3.44c), with provisions for varying r_e, β, $\bar{g}$, μ_i and k_{le}. An option was included so that those terms in the quantities D_{ji} which arise from the right-hand side of boundary condition (3.15a) could be set equal to

zero. Furthermore, either of the particular solutions (3.41) or (3.42) could be substituted into the equation, depending on the temperature distribution.

The cubic equation (3.45) was solved by iteration of Newton's method. Three forms of equation (3.44c) were used in the program: one as given; one with real and imaginary parts written for the case when two of the roots Λ_j of equation (3.48) are complex conjugates; and one with real and imaginary parts for the case when one of the roots Λ_j is negative and $\lambda_j = \sqrt{\Lambda_j}$ is thus imaginary.

The infinite series in the solution was truncated at N terms either when the series was monotonically decreasing and the N-th term of the series was a given percentage of the first term or when N equaled a given maximum number.

A few limitations were imposed on the calculations. When β was less than 0.75, the roots of the cubic equation were so large that terms of the type $e^{\lambda_j i}$ involved in calculation of the quantity Δ_j exceeded the maximum magnitude allowable in FORTRAN. For the same reason, no more than fourteen terms of the infinite series could be calculated.

About fifteen minutes of actual running time on the IBM 709 was required for all of the results obtained.

Discussion of the Results

The results of the calculations discussed in the preceding section are shown in Figures 5 through 8. Each of these graphs shows the dimensionless deflection parameter $\hat{w}$, evaluated at the center of the plate, as a function of the edge loading parameter $k_{/e}$. In Figure 5 the surface of the plate is subject to a double sinusoidal temperature gradient distribution given by

$$R = R_o \sin \pi\xi \sin \pi\eta$$

and the effect of varying the parameter r_e is shown. In Figure 6 the same information is presented for a constant temperature distribution. Figures 7 and 8 illustrate the effect of a varying aspect ratio β on a plate subject to the two temperature gradients.

A basic difference between the solution for a double sinusoidal temperature gradient distribution and for a constant distribution (when there is no transverse load q) is that for the former case the infinite series of the solution (3.44c) reduces to its first term. This fact is seen from the particular solutions (3.42), which are zero for $j > 1$.

From Figures 5 and 6 it is seen that the deflection increases throughout most of the loading process as the

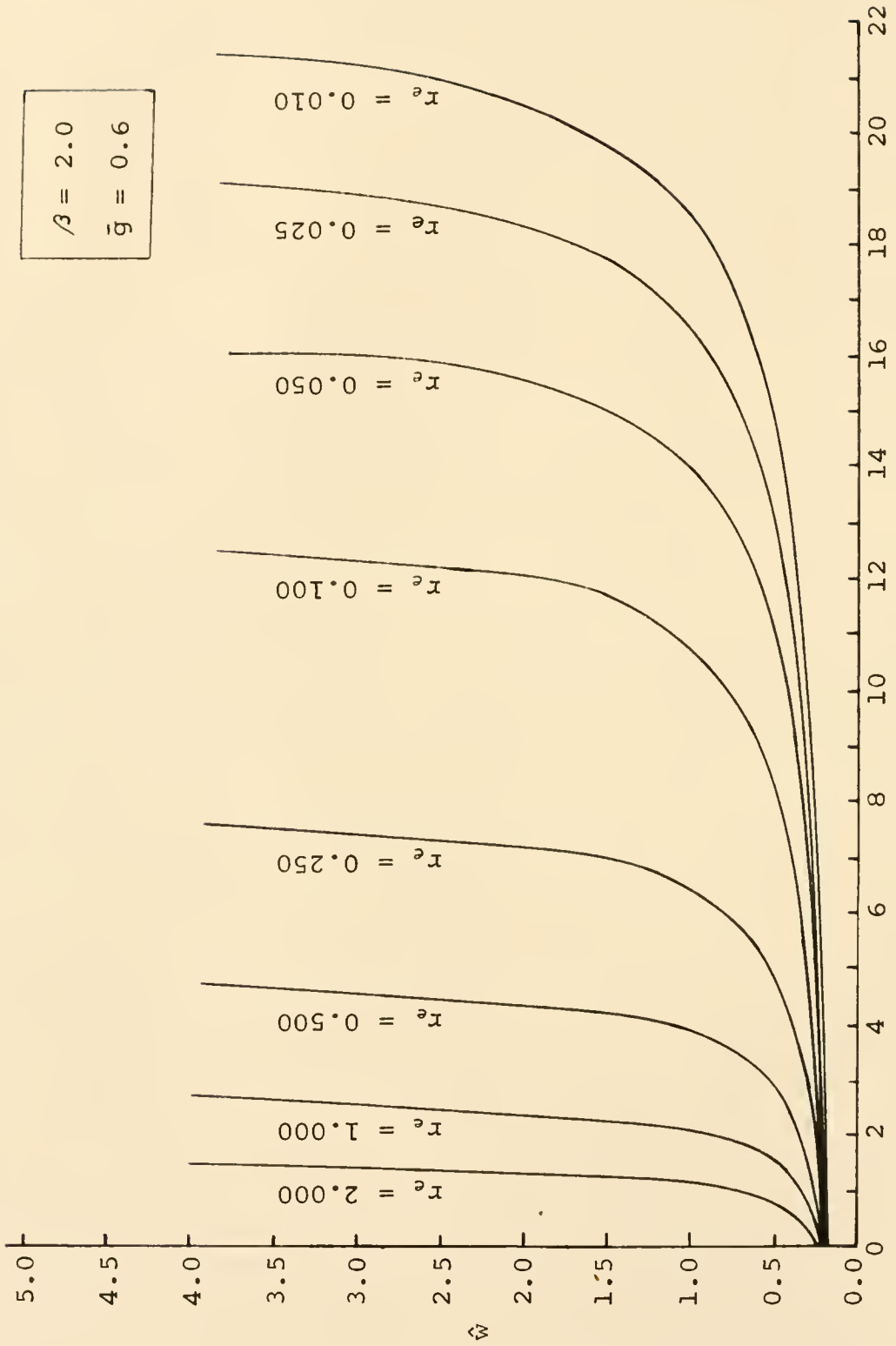

Figure 5. Deflection at the center of a plate under a temperature gradient $R = R_o \sin \pi\xi \sin \pi\eta$ as a function of edge loading

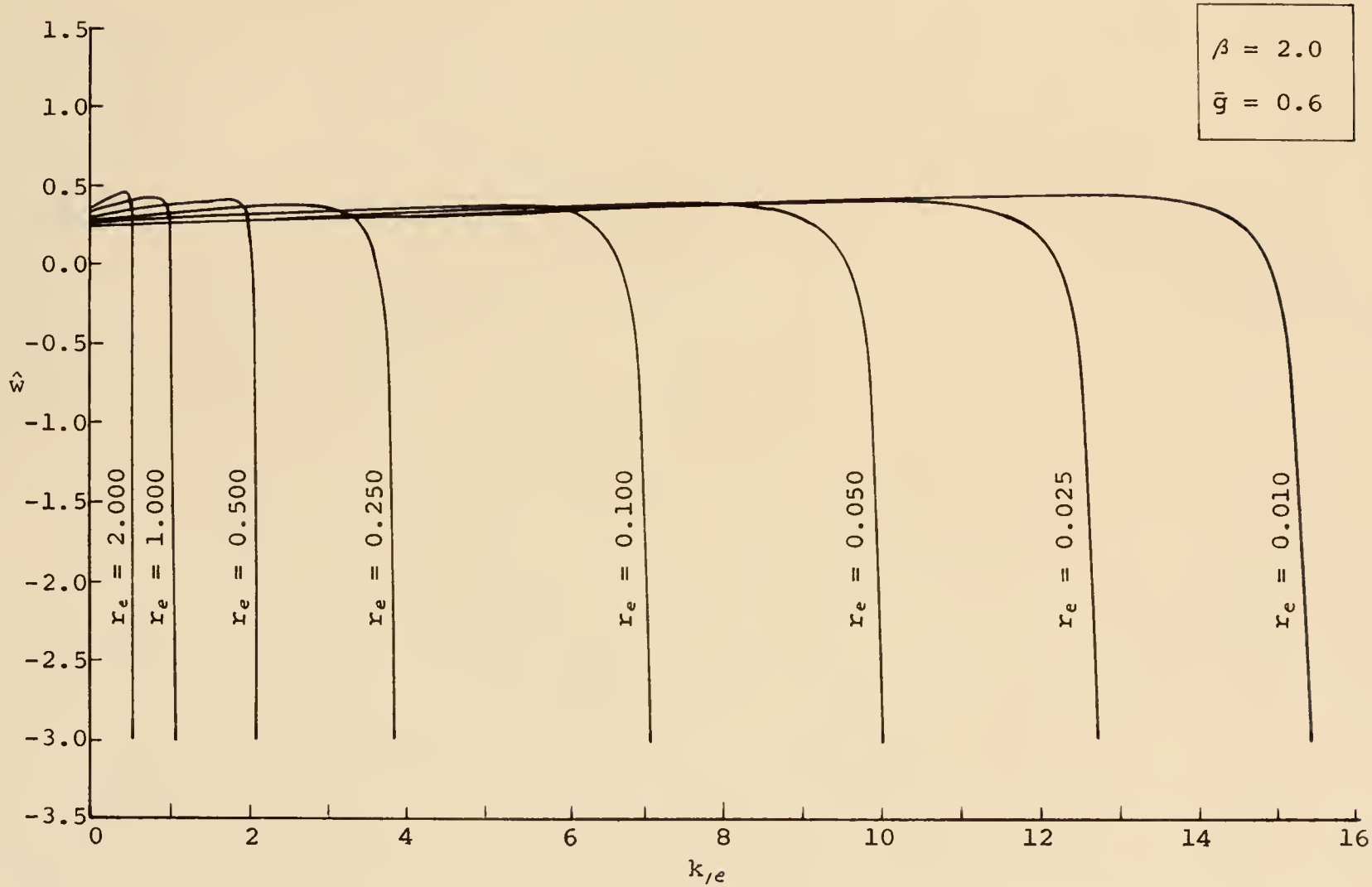

Figure 6. Deflection at the center of a plate under a constant temperature gradient as a function of edge loading

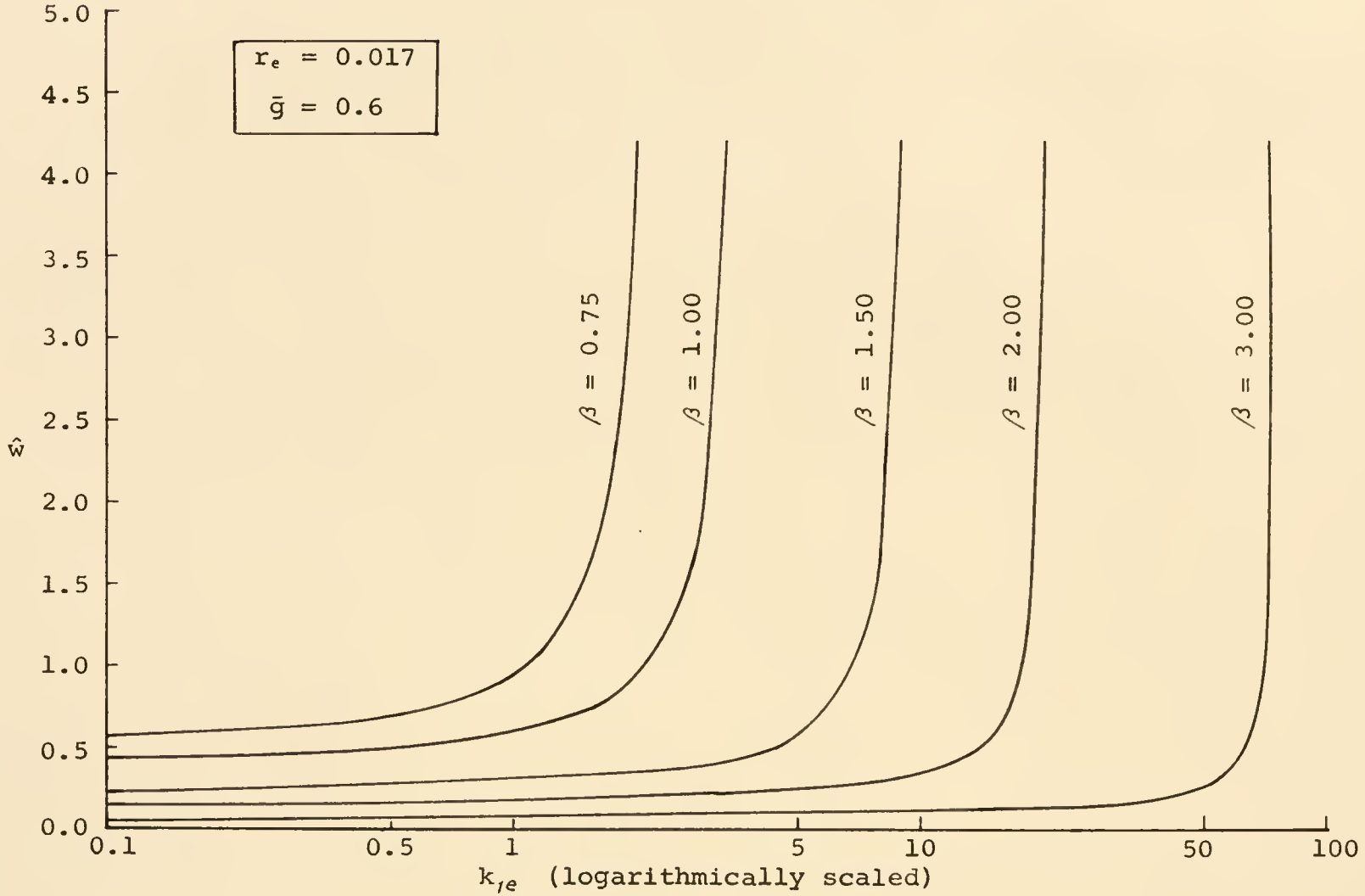

Figure 7. Deflection at the center of a plate under a temperature gradient $R = R_o \sin \pi\xi \sin \pi\eta$ as a function of edge loading

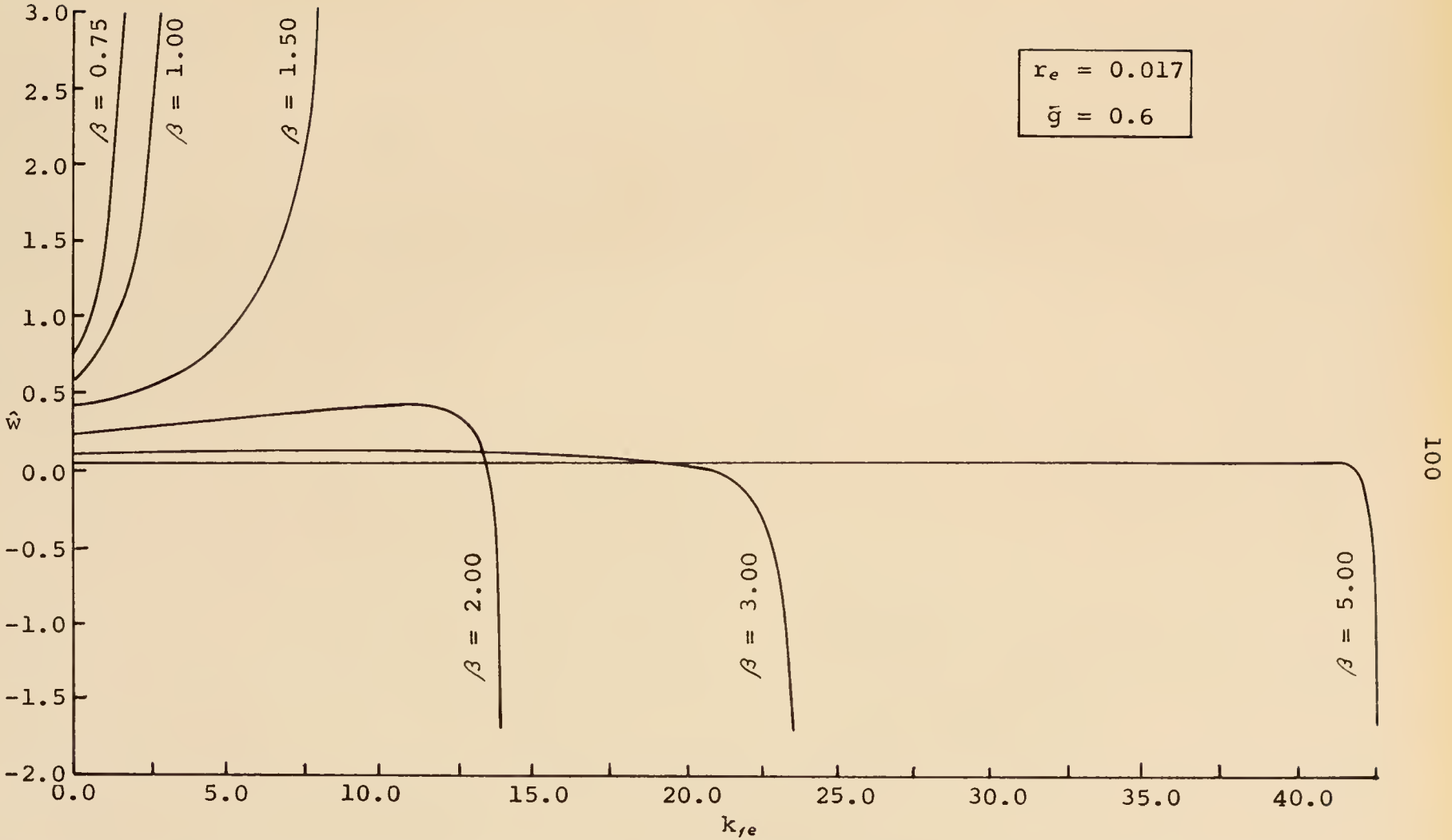

Figure 8. Deflection at the center of a plate under a constant temperature gradient as a function of edge loading

parameter r_e increases, while the buckling load decreases. A marked difference between the two graphs is observed. The deflection monotonically increases until it becomes unbounded and positive at the buckling load in the case of a double sinusoidal temperature gradient distribution; when the distribution is constant, the deflection curve starts to increase for lower values of k_{le}, but then turns downward and decreases until the deflection becomes negatively unbounded at a lower buckling load than for the corresponding case in Figure 5. This effect is known as "reversal of deflections."

The difference in the response of the panel to the two temperature distributions is due to the presence of all the terms in the infinite series for the second case, as mentioned above. Buckling occurs at the lowest value of k_{le} so that the denominator Δ_j of a term in the series approaches zero. The lowest such value of k_{le} need not cause the denominator in the first term of the series to vanish; the buckling load may be determined by any term in the series, depending on the properties of the plate. This phenomenon occurs in homogeneous thin plate theory also.[25] In order to describe this effect, the term "n-th mode buckling" is introduced, meaning that the critical load occurs when the denominator of the series term with index j = n vanishes.

From another point of view, if the external forces acting on the plate were due only to the compressive edge loading and no temperature gradient were applied, then the problem would become the associated eigenvalue problem of the one studied here; in this case the critical load would cause the plate to buckle into the n-th mode shape, or into n half-waves in the ξ-direction. When $n > 1$, the effect is denoted as "higher mode buckling." The buckling load is always of the higher mode type when $\beta = 2.0$, $\bar{g} = 0.6$, and r_e varies from 0.01 through 2.00, as seen in Figures 5 and 6.

In the evaluation of equation (3.44c) on the computer it was found invariably that each term in the series alternated in sign. This result is not unexpected because of the presence in the solution of the factor $\sin j\pi\eta$, where $\eta = 1/2$ at the center of the plate; however, the sign of the remaining part of the solution could not be determined beforehand for the general case. It is interesting to observe in Figure 6 that whenever higher mode buckling occurred, the mode involved had an index of $j = 3, 7, 11, \ldots$, so that the term had a negative sign and thus the deflection always became negatively unbounded.

The deflection at the center of the plate varies similarly when r_e is held constant and β is varied. As β

increases, the deflection for a given value of k_{fe} decreases and the buckling load increases, as is shown in Figures 7 and 8. From Figure 8 it is seen that the values of the parameters chosen permit first-mode buckling for a range of values of β. Once again the heuristic result is noted that higher mode buckling was characterized by a negative value of deflection.

One of the results of the present analysis that should be useful is the calculation of more accurate values of the buckling load. When specific values of the properties and dimensions are known, the buckling load can be found by evaluation on the computer. Unfortunately, the expression (3.46) for Δ_j is so cumbersome that an analytical solution for the buckling load is forbiddingly difficult. However, the deflections and buckling loads were found to agree closely with Ebcioglu's results.[23] His expression for the buckling load is

$$k_{fe} = \left\{ r_e(1-\mu)\left[\beta^6\bar{g} + \beta^4 j^2(1+2\bar{g}) + \beta^2 j^2(2+\bar{g}) + j^6\right] + 2\bar{g}(\beta^2 + j^2)^2\right\} \Big/ j^2\left[r_e^2(1-\mu)(j^2+\beta^2)^2 + r_e\left\{\beta^2\left[2+\bar{g}(1-\mu)\right] + j^2\left[2\bar{g} + (1-\mu)\right]\right\} + 2\bar{g}\right] \tag{5.1}$$

A comparison of the deflection calculated from Ebcioglu's results with the deflection calculated from the present results is shown in Figure 9. Table 3 shows the comparison of the buckling loads. The error is never greater than 10 per cent. Therefore, it is felt that using equation (5.1) for approximate calculations of the buckling load and its behavior as a function of r_e, β, and j is justified.

TABLE 3

COMPARISON OF BUCKLING LOAD VALUES

r_e	β	k_{fe} at Buckling	
		Ebcioglu's Results	Present Results
0.025	2.0	13.5	12.9
0.050	2.0	10.6	10.1
0.100	2.0	7.41	7.1
0.250	2.0	3.85	3.9
0.500	2.0	2.00	2.0
1.000	2.0	1.00	1.0
2.000	2.0	0.50	0.5
0.017	0.75	2.36	2.4
0.017	1.00	3.83	3.6
0.017	2.00	14.8	14.1
0.017	3.00	25.6	23.8
0.017	5.00	43.5	42.7
0.017	10.00	58.9	59.4

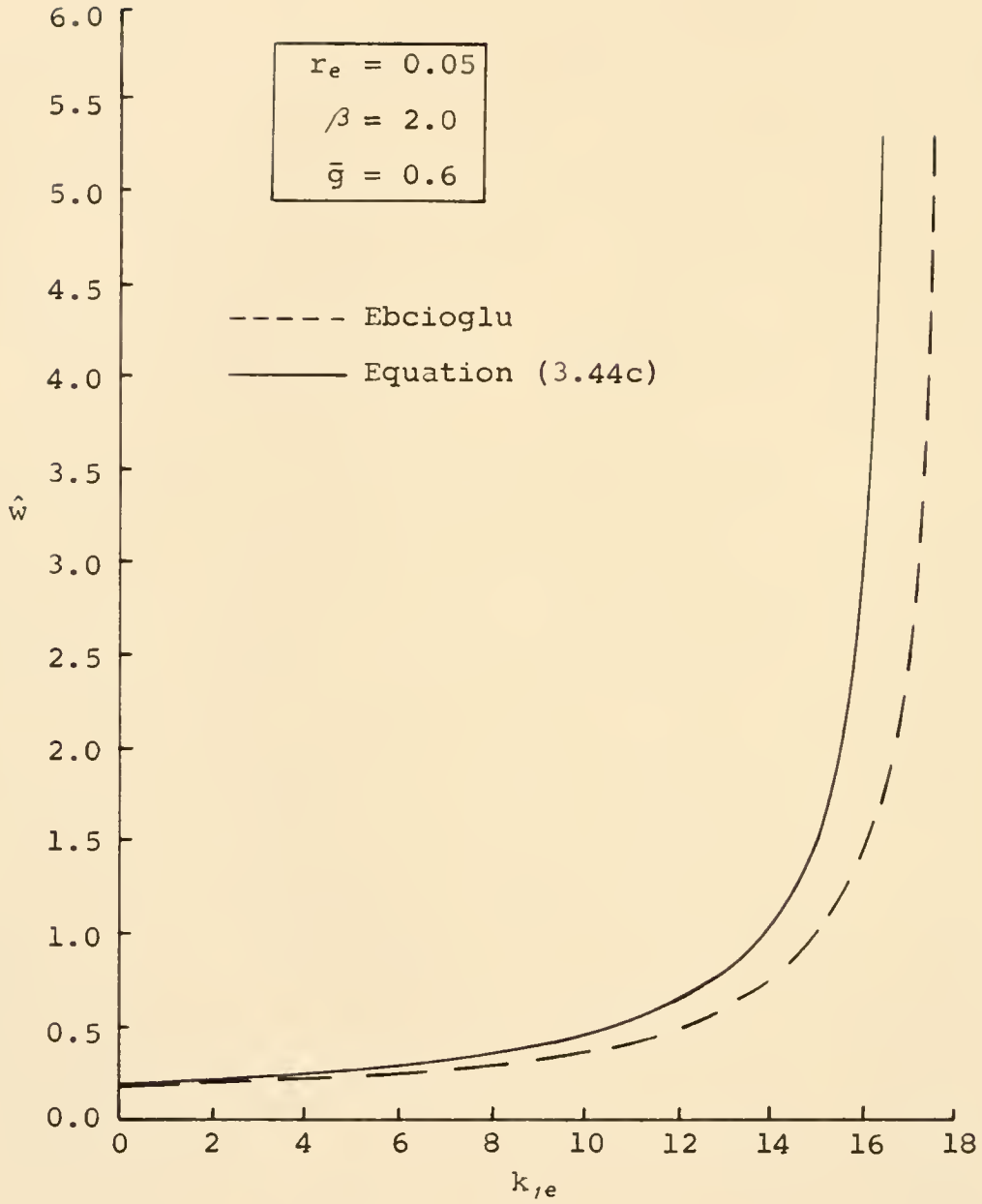

Figure 9. Comparison of the deflection at the center of a plate under a temperature gradient $R = R_o \sin \pi\xi \sin \pi\eta$ as a function of edge loading

The buckling mode is determined by the equation

$$\frac{dk_{,e}}{dj} = 0 \tag{5.2}$$

where $k_{,e}$ is given by expression (5.1). This equation locates the value of j at which the smallest critical value of $k_{,e}$ occurs. Since j must be an integer, this equation may only produce an approximation of the exact value of j.

It is interesting to note that cases exist where there are no real roots of equation (5.2). When (5.2) is written in full, a quartic equation results. If, for example, $\beta = 2.0$, $\bar{g} = 0.6$, and $r_e > 0.495$, all coefficients of this equation have like signs, and it follows from Descartes' rule of signs that there are no positive values of j satisfying this equation. Therefore, the function $k_{,e}$ is monotonic for $j > 0$, and a simple check at $j = 0$ and $j \to \infty$ verifies that $k_{,e}$ monotonically decreases to its asymptotic value

$$\lim_{j \to \infty} k_{,e} = \frac{1}{r_e}$$

It follows that the critical value of $k_{,e}$ is determined by the mode corresponding to $j \to \infty$. As a result, the sign of the deflection as $k_{,e}$ approaches the critical load becomes indeterminate.

The explanation of this effect is that as the number of buckling waves increases, the bending stresses in the faces will dominate the membrane stresses. However, terms due to bending effects were neglected in the derivation of equations (3.44c) and (5.1). Finite roots should result if these terms are retained.

As a result of the preceding discussion, it is necessary to make a qualifying remark about the deflection curve in Figure 6 for r_e = 0.5, 1.0, and 2.0. According to this discussion, the portions of the curves near the buckling load should be indeterminate. The values shown were actually calculated from an infinite series truncated at the term j = 23.

CHAPTER VI

SUMMARY

The equations governing the deflections of a sandwich panel subject to a general system of loads and a thermal gradient were developed and solved for the special case of uniaxial loading. Application of a Levy-type solution enabled the problem to be solved without making simplifying assumptions about the boundary conditions. The solution for the vertical deflection was evaluated on a digital computer.

It was seen that when bending stresses in the faces are neglected, the response of the panel is determined by three nondimensional parameters. The buckling loads were seen to increase as the aspect ratio increased and to decrease as the parameter r_e increased. Reversal of deflection generally occurred. It was noted that buckling is characterized by a large number of half-waves for panels with properties and dimensions commonly used in practice. An unusual effect of an indeterminate buckling mode was noted and accounted for by the fact that the bending stresses should be included for high buckling modes.

APPENDIX

The dimensions of the sandwich panels used by Ebcioglu et al.[24] and the eight points on the faces at which the temperature was measured are shown in Figure 10.

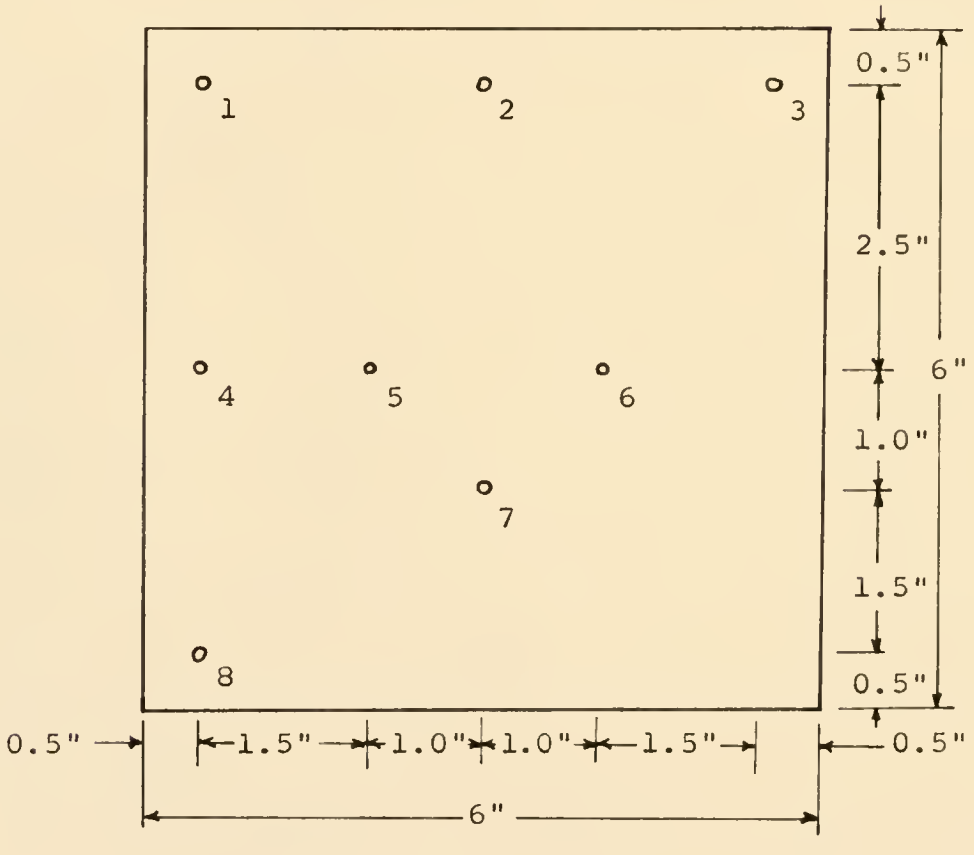

Figure 10. Dimensions and data points of the sandwich panel

The temperature gradients that were measured are given in Table 4.

TABLE 4

EXPERIMENTAL TEMPERATURE GRADIENT

Data Point Number	Temperature Gradient (^{o}F)			
	Trial 1	Trial 2	Trial 3	Trial 4
1	12	12	16	18
2	16	19	21	33
3	14	19	19	30
4	35	37	39	58
5	53	56	60	97
6	53	59	63	97
7	48	51	58	80
8	26	23	21	30

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BIOGRAPHICAL SKETCH

Edward Linde Bernstein was born in Newark, New Jersey, on February 9, 1938. He spent the early part of his life in the East and later moved to Chicago, where he received his secondary education. He entered Purdue University in the fall of 1955 and received a Bachelor of Science in the field of Engineering Sciences in June, 1959. He then enrolled at Cornell University in the Engineering Physics curriculum; he received a Master of Science in this area in September, 1961, having held a Cornell Graduate Fellowship and a teaching assistantship. Immediately following the granting of this degree, he entered the University of Florida to pursue studies leading to the degree of Doctor of Philosophy. He has held the positions of Assistant in Research, NSF Cooperative Fellow, and NASA Pre-Doctoral Trainee. He is currently a Research Associate.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Engineering and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

April 18, 1964

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